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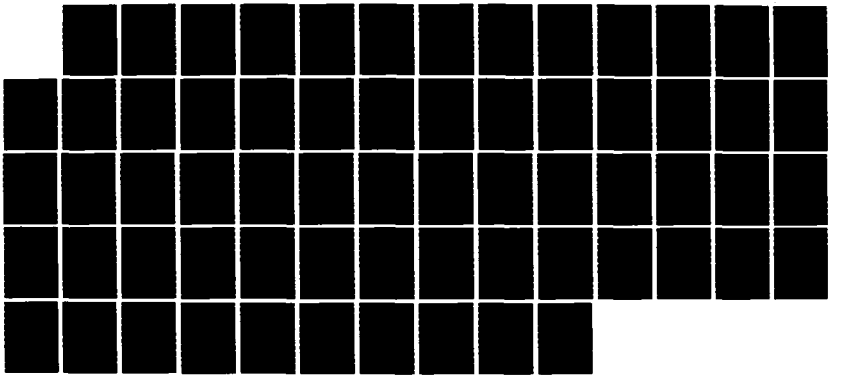

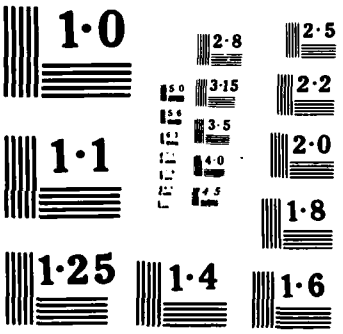
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Department of Statistics
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A Central Limit Theorem for Markov Paths

And Some Properties of Gaussian Random Fields

by

Robert J. Adler and R. Epstein

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This central limit result, together with related results indicating how additive functionals of the Markov processes generate additive functionals of the fields, yield

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Although the paper is comprised primarily of new results, and despite the fact that the subject matter is somewhat esoteric, our aims are primarily didactic and expository - we want to try to initiate the uninitiated into some of the mysteries of generalised processes via an easily understood model.

A Central Limit Theorem for Markov Paths And Some Properties of Gaussian Random Fields

by

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Abstract

Our primary aim is to "build" versions of generalised Gaussian processes from simple, elementary components in such a way that as many as possible of the esoteric properties of these elusive objects become intuitive. For generalised Gaussian processes, or fields, indexed by smooth functions or measures on $\mathbf{R}^d$, our building blocks will be simple Markov processes whose state space is $\mathbf{R}^1$. Roughly speaking, by summing functions of the local times of the Markov processes we shall, via a central limit theorem type of result, obtain the Gaussian field.

This central limit result, together with related results indicating how additive functionals of the Markov processes generate additive functionals of the fields, yield considerable insight into properties of generalised Gaussian processes such as Markovianess, self-similarity, "locality" of functionals, etc.

Although the paper is comprised primarily of new results, and despite the fact that the subject matter is somewhat esoteric, our aims are primarily didactic and expository - we want to try to initiate the uninitiated into some of the mysteries of generalised processes via an easily understood model.

1. Introduction.

This paper is about generalised Gaussian processes and some of their properties, or, to be more precise, it is about a way to think about these things without becoming too confused. Generalised Gaussian processes can be succinctly defined as isometric mappings from a Sobolev space to a space of Gaussian variables, and virtually all their interesting properties can be stated in terms of the associated Fock space. While such definitions and statements are both neat and precise they do nothing to help the non-expert understand what these rather esoteric objects are. Our plan is to write for the novice – we shall start at a point which seems to us to be a natural beginning, and follow a natural sequence of observations and constructions that will ultimately show Sobolev and Fock spaces, along with much that goes with them, in a fashion that is intuitive for anyone who knows what a Markov process and a central limit theorem (CLT) are.

Since generalised Gaussian processes and all that goes with them are actually a part of Mathematical Physics, (they are "just" Euclidean quantum fields, but we shall take up this point later) and, as such, are supposed to describe the behaviour of elementary particles, let us start by watching such a particle. In fact, let us take $N > 0$ such particles, each one of which, at some common and fixed point of time, "pops" into existence and then proceeds to wander about space, which we take to be $\mathbf{R}^d$, $d \geq 2$. Assume the "birthplaces" of these N particles, $x_1, \ldots, x_N$ are distributed randomly in $\mathbf{R}^d$ according to some law, and that each particle executes a Markov motion through $\mathbf{R}^d$, described by N independent Markov processes $X_1(t), \ldots, X_N(t)$, $t > 0$. To make our story sound more like Physics, we shall also assume that each of these particles begins its life with a charge σ_i, where $P(\sigma_i = +1) = P(\sigma_i = -1) = 1/2$. The charges of different particles are assumed independent.

To complete the story, suppose that at time t the original unit charge has dissipated by a factor of e^{-t}, and that as each particles performs its Markov motion through $\mathbf{R}^d$ it leaves each point it hits with a charge equal to its own charge at the time of hitting (i.e. σe^{-t}). What we, as amateur physicists, would like to know is what does the charge distribution

throughout $\mathbf{R}^d$ look like after all our particles have decayed to inconsequentiality. Clearly, as probabilists, we shall only be interested in the $N \to \infty$ limit of this question!

Considering only one such particle for the moment, it is clear that the key to solving our problem must lie in the definition of some sort of weighted *local time*, a random function $L_x(X)$ that would measure the amount of time the path $X(t), t > 0$, spends at the point $x \in \mathbf{R}^d$, weighted by its charge when visiting. That is, we would like to make sense of the equation

$$L_x(X) = \int_0^\infty e^{-t} \delta(X(t) - x) dt, \tag{1.1}$$

where δ is the Dirac delta function. For the moment let us take (1.1) at face value, even though, as we shall see later, it has no fully rigourous meaning. Then our problem reduces to studying the behaviour of the sum

$$\Phi_N(x) := N^{-1/2} \sum_{k=1}^{N} \sigma_k L_x(X_k)$$

as $N \to \infty$. If there are enough moments about (and we shall ensure that there are) then the independence of the X_k should, with a standard CLT, ensure that the Φ_N converge to a Gaussian random field on $\mathbf{R}^d$. This would be a first step towards solving our problem. (But only a *first* step, since, as amateur physicists, we would also like to know something about how particles interact. So far, nothing in our model allows for interaction.)

Unfortunately, this simple minded approach has one basic flaw in it - the integral in (1.1) does not generally exist. Probabilistically this is because most Markov processes (e.g. Brownian motion) do not hit points in $\mathbf{R}^d$ $d \geq 2$, with non-zero probability. (In terms of the physics of the limit random field this turns out to be related to such elegant concepts as the Heisenberg uncertainty principle!) To get around this difficulty, we note that instead of asking how much weighted time each particle spends at a given point, we could ask about the time it spends in some small set A. Writing 1_A for the indicator function of A, this would lead us to a weighted *occupation measure* which we could denote by

$$L_A(X) = L(1_A; X) := \int_0^\infty e^{-t} 1_A(X(t)) dt. \tag{1.2}$$

There is no question that this object exists without anything but the mildest measurability conditions on X. Of course, if (1.1) were justifiable, then we would have

$$L(1_A; X) = \int_A L_x(X)dx = \int_{\mathbf{R}^d} 1_A(x)L_x(X)dx.$$

Freely generalising these equivalences, let us now take f to be any "nice" (we shall be more specific later) function on $\mathbf{R}^d$, and consider a time and space weighted local time of the form

$$L(f; X) := \int_{\mathbf{R}^d} f(x)L_x(X)dx. \tag{1.3}$$

With f an indicator function we recover the well-defined (1.2). With f a delta function we recover the ill-defined (1.1). With f belonging to just the right class of functions (Schwartz or Sobolev space – but wait for details) (1.3) can be shown to make perfect sense. Another way to write (1.3) is to notice that if, in fact, L_x existed, then we would have

$$L(f; X) = \int_0^\infty e^{-t} f(X(t))dt. \tag{1.4}$$

In this formulation $L(f; X)$ is certainly well-defined with only minimal assumptions on f. Nevertheless, (1.4) is undesirable for two reasons. Firstly, it does not generalise easily to the measure-indexed local times we shall consider in a moment, and, secondly, it does not have the intuitive appeal of (1.3).

If, in (1.3), we think of f as being a weighting function for the point indexed local time, then it is only a small step to think of weighting by measures. That is, if μ is a nice measure on $\mathbf{R}^d$ then we can make sense of the weighted local time $L(\mu, X)$ which we write, symbolically, as

$$L(\mu; X) := \int_{\mathbf{R}^d} L_x(X)\mu(dx).$$

Again, although L_x may not exist, it is possible to make sense out of $L(\mu, X)$.

Returning now to our CLT, it is clear that the sums we should really be working with are either

$$\Phi_N(f) := N^{-1/2} \sum_{k=1}^{N} \sigma_k L(f, X_k) \tag{1.5}$$

or

$$\Phi_N(\mu) := N^{-1/2} \sum_{k=1}^{N} \sigma_k L(\mu, X_k). \tag{1.6}$$

The limits of these sums will be Gaussian variables parameterised, as we vary the weights f and μ, by either a family of functions or a family of measures. The Gaussian processes thus obtained will be the generalised Gaussian processes that we mentioned in our opening sentence, and the CLT described will be our tool for studying them. This, however, does not represent the end of our task, for we have not yet found a way to describe interactions between particles.

For the sake of simplicity, let us consider for the moment only a very simple type of interaction. Recall that as each of our particles passes a point $x \in \mathbb{R}^d$ it leaves behind a charge σe^{-t}, where t is the time at which it hits x. Suppose another particle hits x at time s, leaving behind its charge, and that charge behaves in a multiplicative fashion (a highly unrealistic assumption, but we are only amateur physicists). That is, the charge left at x from the interaction of X_i and X_j hitting at times t_i and t_j, respectively, is $\sigma_i \sigma_j e^{-(t_i+t_j)}$. Consequently, if we are interested in the total interaction charge at x then we must study a new *intersection local time* of the form

$$L_x(X_i, X_j) := \int_0^\infty \int_0^\infty e^{-t} \delta(X_i(t) - x) e^{-s} \delta(X_j(s) - x) dt ds. \tag{1.7}$$

Of course, if (1.1) with its single delta function doesn't exist, then *a fortiori*, neither will this new intersection local time. Nevertheless, we can proceed much as before, firstly replacing the delta function in (1.7) with the indicator function of some set, and then with a general function or measure, to ultimately suggest a weighted intersection local time which could be symbolically written as

$$L(\mu; X_i, X_j) := \int_{\mathbb{R}^d} L_x(X_i, X_j) \mu(dx). \tag{1.8}$$

Working somewhat harder than we have in the above argument, we shall see later that the left hand side of (1.8) can be given a rigourous meaning.

With a notion of intersection strength (between two particles at a time) now defined, we could hope to study a total intersection process by looking at the sum

$$\Psi_N(\mu) := N^{-1}\Sigma\sigma_i\sigma_j L(\mu; X_i, X_j), \tag{1.9}$$

the sum being over all $i < j$, $i, j = 1, \ldots, N$. As probabilists we would now like to send $N \to \infty$ in this expression. It is clear that no simple CLT will work here, since the summands in (1.9) are dependent. Nevertheless, one could hope that since looking at the local times of the Markov paths *themselves lead us to a generalised* Gaussian process, then looking at a functional of these paths (their intersections) should lead us to a functional of the Gaussian limit. In fact, this is precisely what happens, and (1.9) converges to what is known as the ***Wick square*** of the basic Gaussian limit. The Wick square lives in the first interesting part of Fock space, and so we shall now have a way of thinking about generalised Gaussian processes and their Fock spaces in terms of simple Markov particles. This is precisely what we are after!

Before we can go any further, we are going to have to start making the above picture somewhat more precise by defining our terms more carefully. In the following section we shall therefore collect some information on Markov processes and their additive functionals. (Both the local times $L(\mu; X_i)$ and $L(\mu; X_i, X_j)$ are additive functionals.) In Section 3 we shall do the same for generalised Gaussian processes and ***their*** additive functionals. (e.g. Wick squares) In Section 4 we shall then be able to continue where the introduction leaves off, by properly formulating general limit theorems that allow us to describe Gaussian processes and their additive functionals via Markov processes and their functionals. Following this, we consider the insights that these results lead to in Sections 5 and 6.

The casual reader can then stop, for the harder work starts in Section 7 with a proof of special cases of the main result via the method of moments. (This is not too hard.) In order to prove the most general form of the main result, we need to know a little more about the multiple Ito-Wiener integrals that describe Fock space than what is given in Section 3. Section 8 provides this, and also contains the final proof, which is also not too

hard, given that everything has been carefully set up for it. Some closing comments are made in the concluding Section 9.

Before commencing, however, we have some general historic comments and some acknowledgements to make. Our program of viewing Gaussian fields from a Markov outlook is not new. The fields that we shall consider are known to mathematical physicists as Euclidean quantum fields, and are imaginary time versions of real quantum fields. As such they have been studied extensively. Furthermore, it has long been known that they are related to Markov processes, the seminal work in this area being due to Symanzik (1969). This approach has been taken up and developed at length in a program of Dynkin's (1980, 1983, 1984a,b, for example) to relate Gaussian and Markov processes. However, throughout all these projects there is no attempt to tie in the Gaussian and Markov processes via a physical model of the kind described above. The only exception to this is in the work of Wolpert (1978a,b), which builds a model very similar to that we have constructed. In fact, it was Wolpert's elegant construction that lead us to the results that follow. Our results differ from his in two ways. Firstly, they cover a much wider field of models, (he deals only with the so-called "free field" and Brownian motion as the Markov process) and we deal in considerable more depth with the consequences of the model. Secondly, our style of proof is generally quite different, and, we feel much neater.

The main contribution of our paper, then, is not so much the introduction of a completely new idea but rather, the development of a way of looking at things and an exposition of that way that we hope will be at a level to make it available to a wide audience.

Regarding our audience - since a substantial part of our aim has been didactic we have not written this paper for the expert quantum field theorist who already has a well-developed intuition about generalised Gaussian processes. (Although we do hope that even he may find something of interest here.) Rather we have written with the curious novice in mind. In particular, we should point out that most readers will find either Section 2 or Section 3 "well known", although we expect that few will feel that way about both. Feel free to skip the section that you already know. The paper also turned out much

longer than we had ever expected it would. After all, our avowed aim has been to produce something readable, and not a forbidding treatise. However, after a little practice the reader will learn to separate the interesting and important parts from the clumsy and space consuming notations that seem to be endemic to this area of Probability, and the reading of it all should go much faster than one would expect in the beginning.

Finally, the acknowledgements. Our dept to Professor Dynkin's recent work will be obvious by the citations to it throughout the paper. What is not obvious is the debt we owe him for an inspiring set of lectures that he gave during a visit to Israel in 1983 which did much to motivate our work. It is a pleasure to acknowledge that debt now. Haya Kaspi spent a substantial amount of time answering a lot of questions about Markov processes in the early stages of this work. We could not have managed without her. Murad Taqqu helped us to understand parts of Dynkin and Mandelbaum (1983) and Mandelbaum and Taqqu (1984). Most of this paper was written while both authors were visiting the Center for Stochastic Processes in Chapel Hill. The hospitality of the Center and its directors was, as always, gracious and most appreciated.

2. Markov Processes and their Functionals.

2.a. Markov processes and systems.

The treatment of Markov processes and their functionals given in this section comes, essentially, from Dynkin (1981), to which we refer the reader for missing detail.

Let $X(t), t \geq 0$, be a Markov process on a probability space $(\Omega, \mathcal{F}, P)$, taking values in $\mathbf{R}^d, d \geq 1$. We assume the existence of a stationary, symmetric transition density

$$p_t(x, y) = p_t(y, x) \tag{2.1}$$

satisfying $\int_{\mathbf{R}^d} p_t(x, y)dy = 1$ for all x. For each "initial point" x the "probability starting at x" is given by

$$P^x\{X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k\} = \int_{B_1} \cdots \int_{B_k} p_{t_1}(x, dy_1)p_{t_2-t_1}(y_1, dy_2) \cdots p_{t_k-t_{k-1}}(y_{k-1}, dy_k)$$

while each σ-finite initial measure m defines

$$P^m(\bullet) := \int_{\mathbf{R}^d} P^x(\bullet)m(dx). \tag{2.2}$$

For $\theta > 0$ the function

$$g^\theta(x, y) := \int_0^\infty e^{-\theta t} p_t(x, y)dt. \tag{2.3}$$

is called the *Green function* for X or p_t. To obtain a heuristic feel for the Green function, note that for an arbitrary set $A \subset \mathbf{R}^d$

$$\begin{aligned} \int_{\mathbf{R}^d} g^\theta(x, y)1_A(y)dy &= \int_{\mathbf{R}^d} \int_0^\infty e^{-\theta t} p_t(x, y)1_A(y)dydt \\ &= \int_0^\infty e^{-\theta t} P^x\{X_t \in A\}dt \\ &= E^x \int_0^\infty e^{-\theta t} 1_A(X_t)dt. \end{aligned} \tag{2.4}$$

But this, with $\theta = 1$, is the expected value, when X starts at the point x, of the weighted occupation measure of (1.2). Replacing 1_A by a delta function shows that $g^\theta(x, y)$ can

be interpreted as the expectation of the exponentially weighted local time of (1.1), with weighting $e^{-\theta t}$ rather than e^{-t}.

In what follows we shall generally need to consider the *Markov system* of $k > 1$ independent copies $X^1, \ldots, X^k$ of X, which we write as a $\mathbf{R}^{dk}$-valued function on $\mathbf{R}_+^k$ via

$$\mathbf{X}(\mathbf{t}) := (X^1(t^1), \ldots, X^k(t^k)). \tag{2.5}$$

For notational convenience, we introduce k-dimensional "transition" and "Green" functions as

$$p_{\mathbf{t}}(\mathbf{x}, \mathbf{y}) := \prod_{i=1}^{k} p_{t^i}(x^i, y^i), \quad g^\theta(\mathbf{x}, \mathbf{y}) := \prod_{i=1}^{k} g^\theta(x^i, y^i),$$

with $\mathbf{x} = (x^1, \ldots, x^k) \in \mathbf{R}^{dk}$. Note that by (2.1) we always have $g^\theta(\mathbf{x}, \mathbf{y}) = g^\theta(\mathbf{y}, \mathbf{x})$.

Throughout the remainder of the paper we shall assume that all processes are ***right*** in the sense of Dynkin (1981).

2.b. Additive functionals.

To start with an easy case, let X be a Markov process in $\mathbf{R}^d$ and $\mathbf{X}$ a system of k independent copies of it. Let b be a positive, bounded, measurable function on $\mathbf{R}^{dk}$ and $\theta > 0$. Then the path integral

$$F^\theta(I) := \int_I e^{-\theta(t_1 + \ldots + t_k)} b(\mathbf{X}(\mathbf{t}))\, d\mathbf{t} \tag{2.6}$$

defines a family of random variables, indexed by open intervals $I \subseteq \mathbf{R}_+^k$, with the following properties:

(2.7) For every path, the function F^θ can be extended to a measure on $\mathbf{R}_+^k$,

(2.8) For every I, $F^\theta(I)$ is an element of the minimal σ-algebra generated by the $\mathbf{X}(\mathbf{t})$, $\mathbf{t} \in I$,

(2.9) A shift $\mathbf{t} \to \mathbf{s} + \mathbf{t}$ of the path of $\mathbf{X}$ induces an analogous shift of the corresponding measure.

The fact that we have written the exponential term in (2.6) explicitly, and not as part of the function b, is to recall the exponential weakening of the Introduction. However, in

one form or another, some damping is necessary for the integral to be finite. (Even the simple case $b = 1_A$ would lead to a divergent integral if X is neighbourhood recurrent, unless the exponential is included.) Rather than damping X, we could leave it untampered until some exponential killing time. This would lead to a similar theory and, in terms of the Introduction, would take us closer to Wolpert's model. Purely for reasons of mathematical convenience, we prefer to dampen in a continuous fashion.

Any family with the properties (2.7)-(2.9) is called a ***homogeneous additive functional*** of $\mathbf{X}$. The class of all such functionals is much larger than that obtained by integrals of the form (2.6). In fact, what we shall be most interested in will be situations in which $b(\mathbf{X}(\mathbf{t}))$ "$=$" $\delta^{k-1}(X^1(\mathbf{t}) - X^2(\mathbf{t}), \ldots, X^{k-1}(\mathbf{t}) - X^k(\mathbf{t}))$, with δ^m the Dirac delta function on $\mathbf{R}^{md}$. In such a case, the additive functional F^θ measures the set of common points of $X^1, \ldots, X^k$, in a sense to become clearer later.

To extend (2.6), let γ, ν be measures on $\mathbf{R}^{dk}$, and introduce the inner products

$$
\begin{aligned}
(\gamma, \nu)_\theta &= \int \gamma(d\mathbf{x}) g^\theta(\mathbf{x}, \mathbf{y}) \nu(d\mathbf{y}) \\
&= \int \gamma(dx^1, \ldots, dx^k) g^\theta(x^1, y^1) \ldots g^\theta(x^k, y^k) \nu(dy^1, \ldots, dy^k).
\end{aligned} \tag{2.10}
$$

Let $\mathcal{M}^{\theta,k} = \mathcal{M}^k(g^\theta)$ denote the set of all σ-finite γ for which $\langle \gamma, \gamma \rangle_\theta < \infty$. Also, for $\mathbf{s}, \mathbf{t} \in \mathbf{R}^k$ write $\mathbf{s} < \mathbf{t}$ if $s^i < t^i$ for all i, and if $\mathbf{s} < \mathbf{t}$ then let $(\mathbf{s}, \mathbf{t})$ denote the open interval $\{\mathbf{u} : s^i < u^i < t^i, i = 1, \ldots, k\}$. Then the following theorem, linking measures in $\mathcal{M}^{\theta,k}$ and additive functionals, is a trivial extension of Dynkin (1981). (Dynkin treats the case $\theta \equiv 1$. The extension to general θ is immediate.) For notational convenience, we write $|\mathbf{t}|$ for $t_1 + \ldots + t_k$, and $||\mathbf{t}||^2$ for $t_1^2 + \ldots + t_k^2$.

Theorem 2.1. *Let the k components of $\mathbf{X}$ be independent copies of a symmetric right process with initial measure m = Lebesgue measure. Then to every measure $\gamma \in \mathcal{M}^{\theta,k}$ there corresponds an additive functional F^θ_γ of $\mathbf{X}$ with the following properties:*

I: There exists a negligible set Ω_0 such that for all $w \notin \Omega_0$ $F^\theta_\gamma(w; (\mathbf{s}, \mathbf{t}))$ is finite for all $0 < \mathbf{s} < \mathbf{t}$ and is continuous in $\mathbf{s}$ and $\mathbf{t}$.

II: For every positive, Borel $f : \mathbf{R}_+^k \times \mathbf{R}^{dk} \to \mathbf{R}$

$$E^m \int_{\mathbf{R}_+^k} f(\mathbf{t}, \mathbf{X}(\mathbf{t})) F_\gamma^\theta(d\mathbf{t}) = \int_{\mathbf{R}_+^k} \int_{\mathbf{R}^{dk}} e^{-\theta|\mathbf{t}|} f(\mathbf{t}, \mathbf{y}) \gamma(d\mathbf{y}) d\mathbf{t} \tag{2.11}$$

Whereas Theorem 2.1 guarantees us an F_γ^θ for each $\gamma \in \mathcal{M}^{\theta,k}$, one can also go the other way. That is, given an additive functional F of $\mathbf{X}$, the so-called ***spectral measure*** γ_F defined by

$$\gamma_F(A) := \theta E^m \int 1_A(\mathbf{X}(\mathbf{t})) F(d\mathbf{t})$$

$(A \subset \mathbf{R}^{dk})$ defines a measure satisfying (2.11).

For a measure $\gamma \in \mathcal{M}^{\theta,k}$ the corresponding additive functional F_γ^θ can be constructed in one of two ways. If γ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^{dk}$, so that $\gamma(d\mathbf{x}) = b(\mathbf{x})d\mathbf{x}$ with $\int b(\mathbf{x}) g^\theta(\mathbf{x}, \mathbf{y}) b(\mathbf{y}) d\mathbf{x} d\mathbf{y} < \infty$, then F_γ^θ can be defined as

$$F_\gamma^\theta(I) = \int_I e^{-\theta|\mathbf{t}|} b(\mathbf{X}(\mathbf{t})) d\mathbf{t}, \quad I \subseteq \mathbf{R}_+^k, \tag{2.12}$$

so that we return to the form of the simple functional of (2.6). In the general case, F_γ^θ is defined as a limit of path integrals of this form, with the function b being densities of smoothed versions of γ. To be more precise, for $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_k)$, let such a density be given by

$$b_{\gamma,\boldsymbol{\delta}}^\theta(\mathbf{x}) = \int_{\mathbf{R}^{kd}} e^{-\theta|\boldsymbol{\delta}|} p_{\boldsymbol{\delta}}(\mathbf{x}, \mathbf{y}) \gamma(d\mathbf{y}) \tag{2.13}$$

so that the smoothed versions of γ given by

$$\gamma_{\boldsymbol{\delta}}^\theta(d\mathbf{x}) = b_{\gamma,\boldsymbol{\delta}}^\theta(\mathbf{x}) d\mathbf{x}$$

satisfy $\langle \gamma - \gamma_{\boldsymbol{\delta}}^\theta, \gamma - \gamma_{\boldsymbol{\delta}}^\theta \rangle_\theta \to 0$ as $||\boldsymbol{\delta}|| \to 0$. Then if γ is finite (i.e. $\gamma(\mathbf{R}^{dk}) < \infty$) the functionals

$$F_{\gamma,\boldsymbol{\delta}}^\theta(I) := \int_I e^{-\theta|\mathbf{t}|} b_{\gamma,\boldsymbol{\delta}}^\theta(\mathbf{X}(\mathbf{t})) d\mathbf{t} \tag{2.14}$$

converge, in $\mathcal{L}^2(P)$, to the functional F_γ^θ of the theorem. If γ is not finite, but belongs to $\mathcal{M}^{\theta,k}$, then there exists a finite measure γ' and a function h such that

$$\gamma(d\mathbf{x}) = h(\mathbf{x})\gamma'(d\mathbf{x}), \qquad \gamma' \in \mathcal{M}^{\theta,k}. \tag{2.15}$$

In this case, F_γ^θ is defined as $F_\gamma^\theta(d\mathbf{t}) = h(\mathbf{X}(\mathbf{t}))F_{\gamma'}^\theta(d\mathbf{t})$.

2.c. An example.

To gain some feeling for the above results, and as a precursor to Section 5, we consider one example. Take $k = d = 2$, $\theta = 1$, and X_1 and X_2 independent Brownian motions in $\mathbf{R}^2$. Then for $x, y \in \mathbf{R}^2$

$$\begin{aligned} g^1(x,y) &= \int_0^\infty e^{-t}(2\pi t)^{-1}\exp\{-\tfrac{1}{2}||x-y||^2/t\} \\ &= cK_0(||x-y||), \end{aligned} \tag{2.16}$$

where K_0 is a modified Bessel function and c an uninteresting constant. For $A \subset \mathbf{R}^2$, take γ_A to be Lebesgue measure on the two-dimensional set D_A in $\mathbf{R}^4$ given by

$$D_A = \{(x_1, x_2, x_3, x_4) : (x_1, x_2) \in A, (x_3, x_4) \in A, x_1 = x_3, x_2 = x_4\}. \tag{2.17}$$

Then, since $K_0(z)$ decays exponentially fast for large z, and has only a logarithmic singularity at $z = 0$, it is easy to check that $\langle \gamma_A, \gamma_A \rangle_1 < \infty$ and so $\gamma_A \in \mathcal{M}^{1,2}$. Thus an $F_{\gamma_A}^1$ satisfying (2.11) exists, and from the form of γ_A and (2.11) it is relatively easy to see that $F_{\gamma_A}^1$ charges only those $\mathbf{t} = (t_1, t_2)$ for which $X_1(t_1) = X_2(t_2)$ and $X_i(t_i) \in A$. The integral $\int F_{\gamma_A}^1(d\mathbf{t})$ now gives us a candidate for the weighted intersection local time of X_1 and X_2 while they are in A. Replacing A by $\mathbf{R}^2$ and γ_A by a measure μ living on $D_{\mathbf{R}^2}$ and satisfying $\langle \mu, \mu \rangle_1 < \infty$ yields

$$L(\mu; X_1, X_2) = \int_{\mathbf{R}^2_+} F_\mu^1(d\mathbf{t})$$

as a properly defined version of (1.8).

Thus, we now have the basic components for the limit theorem discussed in the Introduction, and so we turn to a discussion of what will ultimately yield the limit process.

3. Gaussian Fields and their Functionals.

3.a. Gaussian random fields.

Let $g(x, y)$ be a positive definite function on $\mathbf{R}^d \times \mathbf{R}^d$. Then, in its simplest form, the Gaussian random field with covariance function g is the family of Gaussian random variables $\Phi(x), x \in \mathbf{R}^d$ with zero mean and covariances

$$E\{\Phi(x)\Phi(y)\} = g(x, y). \tag{3.1}$$

(Note the sequence of ideas - for us the covariance function comes first and the random field second.) The random fields that arise in quantum field theory, and the interesting ones that arise as the limit processes mentioned in the Introduction, correspond to positive definite g with $g(x, x) = \infty$ for all $x \in \mathbf{R}^d$, giving the point indexed field $\Phi(x)$ of (3.1) infinite variance and so leaving it poorly defined. The usual way around this is to change the parameter space of Φ to a space of functions. There are two natural choices. One is S_d, the Schwartz space of C^∞ functions that decrease at infinity faster than any polynomial. Given a rich enough probability space $(\Omega, \mathcal{F}, P)$ the Gaussian field on S_d with covariance kernel g is defined as the continuous linear mapping from S_d to the zero mean Gaussian variables in $\mathcal{L}^0(P)$, (with the topology on the latter space being that of convergence in probability), such that the covariance between two values of the field is given via the bilinear functional

$$B_g(f, h) = E\{\Phi(f)\Phi(h)\} = \int\int f(x)g(x, y)h(y)dxdy. \tag{3.2}$$

(If S_d' is the topological dual of S_d - i.e. the space of generalised functions of temperate growth, or, tempered distributions - $\Phi(\bullet)$ can be considered as a S_d'-valued random variable. Hence the now almost archaic term "generalised Gaussian process" used in the Introduction, a term which we now drop in favour of "random field".)

Clearly, if (3.2) is to make sense, we require $B_g(f, f) < \infty$ for all $f \in S_d$, so we are limited in our choice of covariance kernels g to those for which this is true. In order to allow the definition of fields for arbitrary g, we restrict the parameter space to the *Sobolev*

space $S(g)$ of C^∞ functions of finite norm, where the inner product in $S(g)$ is given by $\langle f, h\rangle_g = \int\int f(x)g(x,y)h(y)dxdy$. A Gaussian field on $S(g)$ is defined as for one on S_d, with covariance functional (3.2).

(Note that here $B_g(f,h) = \langle f, h\rangle_g$, leading to our opening comments about Gaussian fields being isometric mappings.)

If a point indexed field satisfying (3.1) exists, then we can always create a S_d or $S(g)$ indexed version of it by setting

$$\Phi(f) = \int f(x)\Phi(x)dx, \tag{3.3}$$

and (3.2) will be satisfied. However, function indexed fields satisfying (3.2) often exist, even when they cannot be represented in the above form.

Another way to index random fields is by the family of measures introduced in the previous section, i.e. $\mathcal{M}^1 = \mathcal{M}^1(g) = \{\mu : \langle \mu, \mu\rangle_g < \infty\}$, where we had

$$\langle \mu, \nu\rangle_g := \int\int \mu(dx)g(x,y)\nu(dy), \tag{3.4}$$

for any two Borel measures μ, ν. The corresponding Gaussian field maps $\mathcal{M}^1(g)$ to zero mean Gaussian variables with $E\{\Phi(\mu)\Phi(\nu)\} = \langle \mu, \nu\rangle_g$. As above, if $\Phi(x)$ exists, we have $\Phi(\mu) = \int \Phi(x)\mu(dx)$. This index set seems to have been initially considered, in the context of the problems that will interest us, by Albeverio and Høegh-Krohn (1979), and was then taken up by Dynkin (1980). Since it turns out to be the most natural parameter space over which to study the Markov property of random fields we shall concentrate on it in the future. However it is important to note that since measures in $\mathcal{M}^1(g)$ can be approximated by elements of S_d or $S(g)$, and vice versa, the theories associated with all three parameter spaces are essentially the same, (except for their Markov properties - for details on this, see Albeverio and Høegh-Krohn (1984) and references therein).

One example of particular interest is the so-called *free field* of mass $m \geq 0$, for which the covariance kernel is given by

$$g_m(x,y) = g_m(x-y) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{e^{ik(x-y)}}{m^2 + ||k||^2}dk. \tag{3.5}$$

Since $g(x,x) = \infty$ for $d \geq 2$, the free field does not exist pointwise, but since the singularities of g are then at worst polynomial, free fields on S_d, $S(g)$ or $M^1(g)$ can be defined.

The kernel (3.5) can be looked upon in a number of ways. It is the kernel of the operator $(-\Delta + m^2)^{-1}$ in $\mathbf{R}^d$, a fact important in Euclidean field theory. More importantly for us, however, is that it is also the Green function g^θ of Brownian motion in $\mathbf{R}^d$ with $\theta = m^2$. This fact provides the link between Gaussian fields and Markov processes discussed in the Introduction.

In fact, if g^θ is the Green function of a symmetric Markov process X, then it is trivial that g^θ is positive definite, so there exists a $S(g^\theta)$ or $M^1(g^\theta)$ indexed Gaussian field with covariance kernel g^θ. These fields are called the fields *associated* with X. Although not every covariance kernel is also a symmetric Green function, this is so often the case that it is worthwhile to study Gaussian fields from the viewpoint of Markov processes.

3b. The spectrum and Wiener-Ito integrals.

For the purposes of this sub-section we shall restrict ourselves to the parameter space S_d. We call a random field *stationary* on S_d if $\Phi(f) \stackrel{\mathcal{L}}{=} \Phi(T_x f)$ for all $f \in S_d$ and $x \in \mathbf{R}^d$, where $T_x f(y) = f(y + x)$. In this case the covariance kernel g of Φ must have the form $g(x,y) = g(x - y)$, and so we write it as a function on $\mathbf{R}^d$ only. Furthermore, (see, e.g. Major (1981) for details) g has inverse Fourier transform G, which is a σ-finite measure on $\mathbf{R}^d$ satisfying $G(A) = G(-A)$ for all $A \in \mathcal{B}^d$ and

$$\int (1 + ||\lambda||)^{-\alpha} G(d\lambda) < \infty \tag{3.6}$$

for some $\alpha > 0$, (i.e. G is *tempered*). We also have

$$E\{\Phi(f)\Phi(h)\} = \int_{\mathbf{R}^d} \hat{f}(\lambda)\hat{h}^*(\lambda) G(d\lambda), \qquad f, h \in S_d,$$

where $\hat{}$ denotes Fourier transformation and $*$ complex conjugation. The measure G is called the *spectral measure* of Φ, and there corresponds to it a random, complex valued, *spectral process* $Z = Z_G$, defined on the same space as Φ, such that

$$\Phi(f) = \int \hat{f}(\lambda) Z_G(d\lambda), \qquad f \in S_d. \tag{3.7}$$

The importance of this representation for us is that not only does it give a simple representation of stationary Gaussian fields on S_d, but it also leads to a neat characterisation of all their $\mathcal{L}^2$ functionals.

To develop this, we define the *Fock space* Γ_Φ (or Γ_G) of the field Φ as the Hilbert space of "Fock columns" $\mathbf{f} = (f_0, f_1, f_2, \ldots)$, where f_0 is a complex number, $f_n = f_n(\lambda_1, \ldots, \lambda_n)$, $n \geq 1$, complex functions of $\lambda_1, \ldots, \lambda_n \in \mathbf{R}^d$, satisfying

$$f_n(\lambda_1, \ldots, \lambda_n) = f_n^*(-\lambda_1, \ldots, -\lambda_n), \tag{3.8a}$$

and such that the norm $||\mathbf{f}||$ admits the expansion

$$||\mathbf{f}||^2 = |f_0|^2 + \sum_{n=1}^{\infty} (n!)^{-1} \int \ldots \int |f_n(\lambda_1, \ldots, \lambda_n)|^2 G(d\lambda_1) \ldots G(d\lambda_n) < \infty. \tag{3.8b}$$

It is well known (e.g. Major (1981)) that there exists a natural unitary mapping (of Hilbert spaces) from Γ_Φ to $\mathcal{L}^2(P)$ defineable via multiple Wiener-Ito integrals, taking $\mathbf{f} \in \Gamma_\Phi$ *to* $\Psi_\mathbf{f}$ *via*

$$\Psi_\mathbf{f} := f_0 + \sum_{n=1}^{\infty} (n!)^{-1} \int \ldots \int f_n(\lambda_1, \ldots, \lambda_n) Z_G(d\lambda_1) \ldots Z_G(d\lambda_n), \tag{3.9}$$

with $||\Psi_\mathbf{f}||^2 = ||\mathbf{f}||^2$. The integrals here are with respect to the spectral measure of (3.7), and do not charge the diagonal $\lambda_1 = \lambda_2 = \ldots = \lambda_n$.

This mapping distinguishes itself from all other unitary mappings $\Gamma_\Phi \to \mathcal{L}^2(P)$ by the fact that the Fock columns with $f_m = 0$ for $m > n$ correspond to polynomials $p(\Phi(h_1), \ldots, \Phi(h_m)), h_i \in S_d$, of degree n in the values of the field. Other important properties of this representation will become clearer later.

3c. Wick powers.

One way to get away from a purely Gaussian theory of random processes while at the same time retaining as much as possible of the inherent analytic simplicity of this case is to work with simple functions, such as powers, of them. In dealing with fields, the corresponding notion is that of a Wick power. For example, if Φ is defined on S_d or $S(g)$

as $\Phi(f) = \int f(x)\Phi(x)dx$, then we would like to define a k-th power $:\Phi^k:$ that would act like a pointwise k-th power of Φ, i.e.

$$:\Phi^k(f) := \int f(x)\Phi^k(x)dx. \tag{3.10}$$

Of course, if (3.3) is ill-defined then so, *a fortiori*, is the above. Nevertheless, there are a number of ways of making mathematics out of (3.10).

The most straightforward would be to consider

$$\lim_{n\to\infty} \int f(x)\Phi^k(\delta_x^n)dx \tag{3.11}$$

as a candidate for $:\Phi^k(f):$, where $\delta_x^1, \delta_x^2, \ldots$, is a sequence of functions in S_d or $S(g)$ converging in some sense to the delta function at x. Note that this approach also works for measure indexed fields. Details of this approach are implicit in Dynkin (1984).

As opposed to the ill-defined (3.10) and the somewhat awkward (3.11), a very neat definition of Wick powers comes by identifying $k!:\Phi^k(f):$ with the coefficient of t^k in the formal power series expansion of $\exp[t\Phi(f) - \frac{1}{2}t^2 B_g(f,f)]$. That is

$$e^{t\Phi(f)-\frac{1}{2}t^2 B_g(f,f)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} :\Phi^k(f): . \tag{3.12}$$

Note that we can replace f here with μ to get measure indexed Wick powers.

Although (3.12) is neat (and useful) it sheds little light on what a Wick power really is.

Since Wick powers belong to $\mathcal{L}^2(P)$ they also have, in the stationary case, a Fock space representation in terms of multiple Wiener-Ito integrals. In fact

$$:\Phi^k(f) := \int \ldots \int \hat{f}(\lambda_1 + \ldots + \lambda_k) Z_G(d\lambda_1) \ldots Z_G(d\lambda_k). \tag{3.13}$$

This representation does not carry over easily to either the non-stationary or the measure-indexed case. Furthermore, although it is tidy, in no way is it obvious why a spectral integral of this form should correspond to the pointwise power (3.10).

It turns out that Wick powers have a very natural representation in terms of the Markov path intersections mentioned in the Introduction, and this sheds light both on their structure and their uses, some of which we shall see in Section 5 below.

3d. Additive functionals.

The last concept that we shall need for the moment relating to Gaussian fields is that of additive functionals. To define these, let $\mathcal{A}_0$ denote set of all closed intervals in $\mathbf{R}^d$ of the form $[a,b] = \{x : a_i \leq x_i \leq b_i\}$, $(a < b)$. Let $\mathcal{A}$ denote the ring of finite unions $A_1 \cup \ldots \cup A_n$ of elements of $\mathcal{A}_0$ with pairwise disjoint interiors, i.e. $\mathring{A}_i \cap \mathring{A}_j = \emptyset$, $i \neq j$.

Write $\mathcal{B}_\Phi$ for the σ-algebra generated by Φ, i.e. $\mathcal{B}_\Phi = \sigma\{\Phi(f)\}$ if the field is S_d or $S(g)$ indexed, or $\mathcal{B}_\Phi = \sigma\{\Phi(\mu)\}$ if the field is $\mathcal{M}^1(g)$ indexed. Also, for open $0 \subset \mathbf{R}^d$ define $\mathcal{B}_{\Phi,0}$ to be the σ-algebra generated by $\Phi(f)$ with $\mathrm{supp} f \subseteq 0$, or $\Phi(\mu)$ with $\mathrm{supp}\mu \subseteq 0$, as the case may be. Then if to every $A \in \mathcal{A}$ there corresponds a random variable $\Psi_A \in \mathcal{L}^2(P)$, measurable with respect to $\mathcal{B}_\Phi$, we call $\Psi : \mathcal{A} \to \mathcal{L}^2(P)$ a ***square integrable functional*** of Φ.

Now, for closed $A \subset \mathbf{R}^d$, set $\mathcal{B}_{\Phi,A} = \bigcap_{\epsilon>0} \mathcal{B}_{\Phi,A^\epsilon}$, where A^ϵ is the ϵ-neighbourhood of A. Then there are three properties of square integrable functionals that will be of interest to us:

I. Locality. For each $A \in \mathcal{A}$ the random variable Ψ_A is measurable with respect to $\mathcal{B}_{\Phi,A}$.

II. Additivity. $\Psi_{A\cup B} = \Psi_A + \Psi_B$, P-a.s for all $A, B \in \mathcal{A}$ with $^\circ A \cap \mathcal{B} = \emptyset$.

III. Stationarity. $\Psi_{A+x} = \hat{U}_x \Psi_A$, where $\hat{U}_x, x \in \mathbf{R}^d$, are the unitary translation operators on $\mathcal{L}^2(P)$ corresponding to the translations $U_x f(y) = f(y-x)$ if $f \in S_d$ or $S(g)$, or to the translations $U_x\mu(A) = \mu(A-x)$ if $\mu \in \mathcal{M}^1(g)$.

The central interest in these three properties lies in the fact that local, additive, stationary functionals can be applied, in a relatively simple fashion, to stationary, Markov, Gaussian fields to yield stationary, Markov *non*-Gaussian fields. The prescription is merely to change from the Gaussian measure P to a new measure $\exp\{\Psi_{\mathbf{R}^d}\} \cdot P$. For details see, for example, Rozanov (1982).

To study these three properties in the case of stationary Gaussian fields, Dobrushin and Kel'bert (1983a,b), noted that since each $\Psi_A \in \mathcal{L}^2(P)$, it must have a Wiener-Ito integral representation of the form (3.9). If we assume, for no loss of generality, that $E\{\Psi_A\} \equiv 0$, then we call the Fock column $(\Psi_1^A, \Psi_2^A, \ldots)$ in this representation the spectral

representation of Ψ_A. Dobrushin and Kel'bert showed that it is possible to read off the above properties from the spectral representation of Ψ. We shall investigate their result in Section 5, via completely different methods, and give it an intuitive explanation not linked to the stationary situation.

We now have (somewhat more than) enough background to look at our first new result.

4. The Limit Theorem and its Consequences.

We now return to the setting of the Introduction, with one small change. Let $(\Omega, \mathcal{F}, P)$ be a probability space rich enough to support the following:

(a) a Poisson random variable $N = N_\lambda$ with mean $\lambda > 0$,

(b) an infinite sequence of independent right Markov processes $X_1(t), X_2(t), \ldots, t \geq 0$, taking values in $\mathbb{R}^d$ and with common stationary, symmetric transition density, $p_t(x, y)$.

(c) an infinite sequence of independent variables $\sigma_1, \sigma_2, \ldots$, with $P(\sigma_i = +1) = P(\sigma_i = -1) = \frac{1}{2}$.

The two infinite sequences and N are all independent of one another.

Fix $\theta > 0$, and let $g^\theta(x, y)$ be the Green function of the X_t. Fix $\lambda > 0$, take $f \in S(g^\theta)$, and define

$$\Phi_\lambda^\theta(f) := (\theta/\lambda)^{\frac{1}{2}} \sum_{i=1}^{N(\lambda)} \sigma_i \int_0^\infty e^{-\theta t} f(X_i(t)) dt. \tag{4.1}$$

This is the type of sum we looked at in (1.5) as being equivalent, in that case, to the ill-defined sum of integrals of local time. Note that, by varying f in (4.1), Φ_λ^θ becomes a random field on Sobolev space. This field has, as $\lambda \to \infty$, a Gaussian limit.

Theorem 4.1. *As $\lambda \to \infty$ the random field Φ_λ^θ converges, in the sense of convergence of finite dimensional distributions, to the zero mean Gaussian field on $S(g^\theta)$ with covariance kernel g^θ.*

This result is actually a special case of the more general Theorem 4.2 below, but it is worthwhile to state it separately because of its simplicity. (Both will be proven in Sections 7 and 8.) The result is also true if, throughout, we replace the Sobolev space $S(g^\theta)$ by the Schwartz space S_d, and impose the extra condition, on g^θ, that

$$\int\int f(x) g^\theta(x, y) f(y) dx dy < \infty, \quad \text{for all} \quad f \in S_d.$$

The extension to fields defined on the space of measures $M^1(g^\theta)$ is not so easy to formulate, since we need to work a little harder defining what we mean by $\Phi_\lambda^\theta(\mu)$. (The Gaussian

limit process is easy to identify, however.) To treat the measure case we now move to a far more general formulation, although, for reasons that we shall note later, we do not yet want to treat the *most* general formulation. Both allow us to consider not just CLT's for the Markov X_t, themselves but also for the interaction of k such processes.

Fix $k \geq 1$, $\theta > 0$, and consider the set of measures $\gamma = \gamma(dx_1, \ldots, dx_k)$ on $\mathbf{R}^{dk}$ that belong to $M^{\theta,k} \equiv M^k(g^\theta)$ and satisfy the symmetry condition

$$\gamma(A_1, \ldots, A_k) = \gamma(A_{i_1}, \ldots, A_{i_k}) \tag{4.2}$$

for each permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$, and $A_i \in \mathcal{B}^d$. We denote this collection by $M^{\theta,k}_{symm}$ or $M^k_{symm}(g^\theta)$.

By Theorem 2.1, there exists an additive functional F^θ_γ corresponding to each $\gamma \in M^{\theta,k}_{symm}$, defined on k of the X_t at a time. Consider firstly the case $k = 1$, and define

$$\Phi^\theta_\lambda(\gamma) := (\theta/\lambda)^{\frac{1}{2}} \sum_{i=1}^{N(\lambda)} \sigma_i F^\theta_\gamma(X_i). \tag{4.3}$$

Then Φ^θ_λ can be considered to be a random field on $M^{\theta,1}_{\text{symm}}$ and an analogue of Theorem 4.1 holds for it.

Theorem 4.2. *As $\lambda \to \infty$ the random field Φ^θ_λ of (4.3) converges, in the sense of convergence of finite dimensional distributions, to the zero mean Gaussian field on $M^{\theta,1}_{symm}$ with covariance kernel g^θ.*

Thus, between Theorems 4.1 and 4.2 we now know how to approximate both function and measure indexed fields via Markov paths. (Note that for a $\mu \in M^{\theta,1}_{symm}$ with density $f \in S(g^\theta)$, we have $\Phi^\theta_\lambda(\mu) \equiv \Phi^\theta_\lambda(f)$, and both theorems identify the same limit random variable.)

We would now like to approximate elements in the $\mathcal{L}^2$ space of a Gaussian field. Since our discussion in the previous section relied on the Fock space representation of the $\mathcal{L}^2$ space, and this required stationarity of the field, we shall assume for the moment that $p_t(x, y) = p_t(x - y)$, or, equivalently, $g^\theta(x, y) = g^\theta(x - y)$. This implies (Blumenthal and

Getoor (1968), p. 17) that the X_i have *independent increments*. We shall lift this condition later, as we shall also do with the following conditions:

(a) $\int\int f(x)g^\theta(x-y)f(y)dxdy < \infty$ for all $f \in \mathcal{S}_d$,

(b) $g^\theta = \hat{G}^\theta$, for some σ-finite, symmetric, tempered measure G^θ on $\mathbf{R}^d$,

(c) $\gamma \in \mathcal{M}^{\theta,k}_{symm}$ has a Fourier transform $\hat{\gamma}$ which exists as a regular (i.e. non-generalised) complex valued function.

We write the set of γ satisfying (c) as ${}^*\mathcal{M}^{\theta,k}_{symm}$.

It follows from (a)–(c), and a generalised Parseval-type equality (c.f. the generalised Poisson summation formula of Argbright and Gil de Lamadrid (1972)) that $\hat{\gamma} \in \mathcal{H}^k_{G^\theta}$, where

$$\mathcal{H}^k_G := \{f = f(\lambda_1,\ldots,\lambda_k), \lambda_i \in \mathbf{R}^d; \|f\|^2_{G\times\ldots\times G} < \infty \quad \text{and}$$
$$f(\lambda_1,\ldots,\lambda_k) = f^*(-\lambda_1,\ldots,-\lambda_k) = f(\lambda_{i_1},\ldots,\lambda_{i_k})\}$$

where

$$\|f\|^2_{G\times\ldots\times G} = \int\ldots\int |f(\lambda_1,\ldots,\lambda_k)|^2 G(d\lambda_1)\ldots G(d\lambda_k).$$

Consequently, if we write Φ^θ for the stationary zero mean Gaussian field on $\mathcal{S}_d$ with spectral distribution function G^θ, then the functional

$$\Psi^\theta_\gamma := \frac{1}{k!}\int\ldots\int \hat{\gamma}(\lambda_1,\ldots,\lambda_k)Z_{G^\theta}(d\lambda_1)\ldots Z_{G^\theta}(d\lambda_k) \tag{4.4}$$

is, according to Section 3b, well defined as an element of the $\mathcal{L}^2$ space of Φ^θ. Furthermore, there corresponds to γ a functional F^θ_γ of k independent X_i. The link between γ, Ψ^θ_γ and F^θ_γ is via the sum

$$\Psi^\theta_\lambda(\gamma) := \begin{cases} 0 & , N(\lambda) < k, \\ (\theta/\lambda)^{k/2} \sum\ldots\sum_{1\le i_1<i_2<\ldots<i_k\le N(\lambda)} \sigma_{i_1}\ldots\sigma_{i_k}F^\theta_\gamma(X_{i_1},\ldots,X_{i_k}) & , N(\lambda) \ge k, \end{cases} \tag{4.5}$$

for which we have

Theorem 4.3. *Under the above conditions, the pair* $\langle\Phi^\theta_\lambda(f), \Psi^\theta_\lambda(\gamma)\rangle$ *converges in distribution, as* $\lambda \to \infty$, *to the pair* $\langle\Phi^\theta(f), \Psi^\theta(\gamma)\rangle$.

Remark. Note that, although we have not stated Theorem 4.3 in such terms, by varying f over $\mathcal{S}_d$ and γ over ${}^*\mathcal{M}^{\theta,k}_{symm}$, we could write the theorem as being about convergence of the

vector processes $\langle \Phi^\theta_\lambda(\cdot), \Psi^\theta_\lambda(\cdot) \rangle$ on $S_d \times {}^*\mathcal{M}^{\theta,k}_{symm}$ to the vector process $\langle \Phi^\theta(\cdot), \Psi^\theta(\cdot) \rangle$, with, as before, convergence being in terms of finite dimensional distributions. This extended form of the theorem is an immediate consequence of the linearity of all the fields on their respective parameter spaces.

Thus, modulo the present technical restrictions on g^θ and γ, we have found a way to represent Gaussian fields via Markov processes, and functionals of the former by functionals of the latter. The technical conditions will all be lifted in Section 8, but their imposition at this stage makes the didactic discussions of the following section somewhat easier to follow. These discussions hinge on using Theorem 4.3 to think of Gaussian fields and their functionals in terms of approximating sums of functionals of Markov processes. However, because of the restrictions we have placed on γ, we cannot consider all functionals of either type. Nevertheless, the next two lemmas show that we can, via these restricted classes of functionals, get arbitrarily close to any functional of interest (in the stationary case). The lemmas and the discussion following them can be skipped on the first reading.

Lemma 4.1. *${}^*\mathcal{M}^{\theta,k}_{symm}$ is everywhere dense in $\mathcal{M}^{\theta,k}_{symm}$.*

Lemma 4.2. *The set of linear combinations of $f \in \mathcal{H}^k_{G^\theta}$ with inverse Fourier transform in $\mathcal{M}^k_{symm}(g^\theta)$ is everywhere dense in $\mathcal{H}^k_{G^\theta}$.*

The consequences of these lemmas, which we shall establish in a moment, are immediate, for, together with Theorem 4.3, they tell us that any Gaussian field on $S(g^\theta)$, together with any of its functionals, has an approximation via Markov processes, (assuming, of course, that the covariance kernel is also the Green function of a right Markov process).

Proof of Lemma 4.1. Consider $\gamma \in \mathcal{M}^{\theta,k}_{symm}$. We shall build a sequence $\gamma_{\boldsymbol{\delta}}, \boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$, with $\gamma_{\boldsymbol{\delta}} \in {}^*\mathcal{M}^{\theta,k}_{symm}$ such that $\gamma_{\boldsymbol{\delta}} \to \gamma$ as $|\boldsymbol{\delta}| \to 0$. Convergence is in terms of the metric defined by (2.10), i.e. via the inner product $(\cdot,\cdot)_\theta$.

Firstly note that by a standard truncation argument the finite $\gamma \in \mathcal{M}^{\theta,k}_{symm}$ are dense in $\mathcal{M}^{\theta,k}_{symm}$, and so we may assume γ to be finite. Now consider the absolutely continuous

approximation to γ used in constructing additive functionals in §2b and given by

$$\begin{aligned}\gamma_{\boldsymbol{\delta}}(d\mathbf{x}) &:= b^{\theta}_{\gamma,\boldsymbol{\delta}}(\mathbf{x})d\mathbf{x} \\ &= \int e^{-\theta|\boldsymbol{\delta}|}p_{\boldsymbol{\delta}}(\mathbf{x},\mathbf{y})\gamma(d\mathbf{y})d\mathbf{x}.\end{aligned}$$

But $b^{\theta}_{\gamma,\boldsymbol{\delta}} \in \mathcal{L}^1(\mathbb{R}^{dk})$, since

$$\begin{aligned}\int |b^{\theta}_{\gamma,\boldsymbol{\delta}}(\mathbf{x})|d\mathbf{x} &= \int d\mathbf{x} \int \gamma(d\mathbf{y})e^{-\theta|\boldsymbol{\delta}|}p_{\boldsymbol{\delta}}(\mathbf{x},\mathbf{y}) \\ &= e^{-\theta|\boldsymbol{\delta}|}\int \gamma(d\mathbf{y}) \int p_{\boldsymbol{\delta}}(\mathbf{x},\mathbf{y})d\mathbf{x} \\ &\leq e^{-\theta|\boldsymbol{\delta}|}\gamma(\mathbb{R}^{dk}) \\ &< \infty.\end{aligned}$$

Thus $b^{\theta}_{\gamma,\boldsymbol{\delta}}$ has a Fourier transform which is a continuous, non-generalised, function and so $\gamma_{\boldsymbol{\delta}} \in {}^*\mathcal{M}^{\theta,k}_{symm}$. Furthermore, it is trivial to check that $\langle \gamma_{\boldsymbol{\delta}} - \gamma, \gamma_{\boldsymbol{\delta}} - \gamma \rangle_{\theta} \to 0$ as $|\boldsymbol{\delta}| \to 0$, so we are done.

Proof of Lemma 4.2.

Let S^c_d denote complex Schwartz space, so that $f \in S^c_d$ is of the form $f = f_1 + if_2$, $f_1, f_2 \in S_d$. Assume, for the moment, that $k = 1$, and note that $\mathcal{H}^1_{G^{\theta}} \subset \mathcal{L}^2(G^{\theta})$. Then since S^c_d is dense in $\mathcal{L}^2(G^{\theta})$ it is easy to see that

$$S^c_{d,symm} := \{f \in S^c_d : f^*(x) = f(-x)\}$$

is also dense in $\mathcal{H}^1_{G^{\theta}}$. Thus, for the case $k = 1$, it will suffice to show that an $f \in S^c_{d,symm}$ can be written as the sum of the Fourier transforms of two measures in $\mathcal{M}^1_{symm}(g^{\theta})$.

To see this, note that the mapping $f \to \widehat{f}$ is an invertible, bicontinuous transformation from S^c_d to itself. Thus if $f \in S^c_{d,symm}$, then $F^{-1}f = \check{f} \in S^c_d$ where F^{-1} represents inverse Fourier transform. Furthermore, $\check{f}$ is real, since

$$\begin{aligned}\check{f}(\lambda) &= \frac{1}{(2\pi)^d}\int \overline{e}^{\iota\lambda x} f(x)dx \\ &= \frac{1}{(2\pi)^d}\int \overline{e}^{\iota\lambda x} f^*(-x)dx \qquad (f^*(x) = f(-x)) \\ &= \frac{1}{(2\pi)^d}\int e^{\iota\lambda x} f^*(x)dx \\ &= (\hat{f}(\lambda))^*.\end{aligned}$$

Writing $h^+ = \max(h, 0)$, $h^- = -\min(h, 0)$, we have, for $f \in S^c_{d,symm}$,

$$(F^{-1}f)(\lambda) = (\check{f}(\lambda))^+ - (\check{f}(\lambda))^-$$

with both $\check{f}^+$ and $\check{f}^-$ real. To obtain measures, write $\gamma_f^+(d\lambda) = \check{f}^+(\lambda)d\lambda$, $\gamma_f^-(d\lambda) = \check{f}^-(\lambda)d\lambda$. Note that by the generalised Poisson summation formula,

$$\begin{aligned}
\infty > & \int |f|^2 G(d\lambda) \\
&= \int f(x)g^\theta(x-y)f(y)dxdy \\
&= \int f^+(x)g^\theta(x-y)f^+(y)dxdy + \int f^-(x)g^\theta(x-y)f^-(y)dxdy \\
&-2\int f^+(x)g^\theta(x-y)f^-(y)dxdy.
\end{aligned}$$

Since $f \in S_d$, and so is arbitrarily small outside large compacts, each of the last three integrals must be finite. Consequently both γ_f^+ and γ_f^- have finite $\langle \cdot, \cdot \rangle_\theta$ norm since, for example, $\langle \gamma_f^+, \gamma_f^+ \rangle_\theta = \int \check{f}^+(x)g^\theta(x-y)\check{f}^+(y)dxdy$. Thus both γ_f^+ and γ_f^- are in $M^1_{symm}(g^\theta)$ and we are done.

The proof for general $k > 1$ follows by noting that the elements of $H^1_{G^\theta} \times \ldots \times H^1_{G^\theta}$ are dense in $H^k_{G^\theta}$ and that for $f \in H^k_{G^\theta}$ of the form $f(x_1, \ldots, x_k) = f_1(x_1) \ldots f_k(x_k)$ with $f_i(x_i) \in H^1_{G^\theta}$ we have $\check{f} = \check{f}_1 \ldots \check{f}_k$. The result for $k = 1$ can then be applied to write $\check{f}$ as a linear combination of measures in $M^k_{symm}(g^\theta)$.

5. Applications, I.

In this, and the following section, we return to what originally motivated us - obtaining insight into random fields via approximating sums of Markov processes. The results range from intuitive, non-rigorous arguments as in our discussion of the Markov property of random fields, (which generate no new results) to rigorous arguments, such as the discussion of renormalisation in the following section where the results are essentially new. We start with what is basically an example for Theorem 4.3, the construction of Wick powers.

5a. Wick powers.

In §3c we gave a sequence of equivalent definitions of Wick powers (see also §8a), including (3.13) which, for $f \in S_d$, set

$$(5.1) \qquad :\Phi^k(f) := \int \dots \int \hat{f}(\lambda_1 + \dots + \lambda_k) Z_{G^\theta}(d\lambda_1) \dots Z_{G^\theta}(d\lambda_k),$$

for Φ with covariance kernel g^θ. Consider the measure

$$(5.2) \qquad \gamma(dx_1, \dots, dx_k) := f(x_1)\delta(x_1 - x_2) \dots \delta(x_{k-1} - x_k)dx_1 \dots dx_k,$$

where δ is the Dirac delta function. (Formally γ is merely the measure concentrated on the diagonal $x_1 = \dots = x_k$ in $(\mathbb{R}^d)^k$, with density $f(x_1)$.) The Fourier transform of γ is given by

$$\begin{aligned} \hat{\gamma}(\lambda_1, \dots, \lambda_k) &= \int e^{i\Sigma\lambda_i x_i} f(x_i)\delta(x_1 - x_2) \dots \delta(x_{k-1} - x_k)dx_1 \dots dx_k \\ &= \int e^{ix\Sigma\lambda_i} f(x)dx \\ &= \hat{f}(\lambda_1 + \dots + \lambda_k). \end{aligned}$$

Thus, comparing with (5.1), we have

$$:\Phi^k(f) := \int \dots \int \hat{\gamma}(\lambda_1, \dots, \lambda_k) Z_{G^\theta}(d\lambda_1) \dots Z_{G^\theta}(d\lambda_k),$$

which is precisely $k!$ times the Ψ^θ_γ of (4.4) and Theorem 4.3.

Consequently, recalling the properties of measures like γ discussed in §2c, we have a representation of Wick powers based on intersections of paths of Markov particles the k-th power coming from intersections of k particle paths.

This representation will work as long as $\gamma \in {}^*\mathcal{M}^{\theta,k}_{symm}$. Clearly, it is only the integrability of γ that is of importance here. If our particles execute Brownian motion, so that the limit Gaussian process is the free field of §3a, it is easy to check that

(5.3) $d = 2 \quad \Rightarrow \quad \gamma \in {}^*\mathcal{M}^{\theta,k}_{symm}$ for all k,

(5.4) $d = 3 \quad \Rightarrow \quad \gamma \in {}^*\mathcal{M}^{\theta,k}_{symm}$ only for $k = 1, 2$,

(5.5) $d \geq 4 \quad \Rightarrow \quad \gamma \in {}^*\mathcal{M}^{\theta,k}_{symm}$ only for $k = 1$.

That is, this technique would seem to work only in very restricted cases. These are precisely the cases in which k independent Brownian motions in $\mathbb{R}^d$ have intersecting paths (c.f. Dvoretsky *et. al.* (1954).)

Nevertheless, if we recall that Gaussian fields also go, in Physics, under the name of Euclidean quantum fields, then we can verify (e.g. Glimm and Jaffe (1982)) that Euclidean field theory turns out to be interesting only in the cases in which our limit theorem applies to Wick powers, so we have not done too badly.

Before leaving this example, we note that it is essentially this problem with $d = 2$ and X = Brownian motion that the two papers of Wolpert (1978a, b) are dedicated to. For more information on the functionals related to F^{θ}_{γ} with γ given by (5.2) with $f = 1_A$, see the papers on *intersection local time* by Geman, Horowitz and Rosen (1984) and Rosen (1985). For more serious applications of this to Physics, see Aizenmann (1982). The original idea of using intersection local time to study Wick powers comes from the seminal paper of Symanzik (1969).

5b. Functionals of Gaussian fields.

In §3d we introduced the notion of functionals of Gaussian fields, along with three possible properties: additivity, locality, and stationarity. Dobrushin and Kel'bert (1983a,b) studied these from the point of view of Fock space, which they used to write a general

functional via its spectral representation. Consider the k-th term in such a representation, Ψ_k^A, defined by

$$\Psi_k^A = \frac{1}{k!} \int \dots \int \upsilon_k^A(\lambda_1, \dots, \lambda_k) Z_G(d\lambda_1) \dots Z_G(d\lambda_k). \tag{5.6}$$

To study the properties of Ψ_k^A as A varies over unions of rectangles, $\mathcal{A}$, let π_k be the natural embedding of the space $\mathcal{H}_G^k$ into the space S'_{dk} of generalised functions and define the mapping $\overline{\pi}_k : \mathcal{A} \to S'_{dk}$ by

$$\overline{\pi}_k(A) = \pi_k(\upsilon_k^A).$$

Then Dobrushin and Kel'bert have shown the following:

I: Ψ_k is ***additive*** iff $\overline{\pi}_k$ is ***additive***, in the sense that

$$\overline{\pi}_k(A_1 \cup A_2) = \overline{\pi}_k(A_1) + \overline{\pi}_k(A_2), \tag{5.7}$$

for disjoint $A_1, A_2 \in \mathcal{A}$.

II: Ψ_k is *local* and ***additive*** iff $\overline{\pi}_k$ is additive and ***diagonal***, in the sense that

$$\text{supp } F^{-1}(\overline{\pi}_k(A)) \subseteq \text{diag}A^k := \{(x_1, \dots, x_k) : x_1 = \dots = x_k \in A\}, \tag{5.8}$$

where F^{-1}, as before, represents inverse Fourier transform.

III: Ψ_k is ***stationary*** iff $F^{-1}\overline{\pi}_k$ is ***invariant*** in the sense that

$$O_{A+x}(f) = O_A(U_x f), \tag{5.9}$$

where $O_A \in S'_{dk}$ is given by $F^{-1}\overline{\pi}_k(A)$, $f \in S_{dk}$, $x \in \text{diag}(\mathbf{R}^d)^k$, and U_x is the shift operator of §3d.

Note two things. Firstly, since everything is based on the Fock space representation, the above results are valid only when the underlying Gaussian field is itself stationary. Secondly, although these results are precise, they have little intuitive appeal. In particular, we ask, why should the inverse Fourier transform of the kernel υ_k^A figure so prominently in the conditions? The answer to this is clear once we approach the problem from the viewpoint of approximating fields and functionals by Markov paths.

We now need to assume that the underlying field has covariance kernel g^θ, which, as usual is a Green function. Consider the sum $\Psi^\theta_\lambda(\gamma)$ of (4.5) with $\gamma = \gamma^A_k$, where $\hat{\gamma}^A_k = \psi^A_k$. (Note, to ensure that ψ^A_k has an inverse Fourier transform in ${}^*\mathcal{M}^{\theta,k}_{symm}$, as γ^A_k must be, we really need to assume this. However, since Lemma 4.2 implies that such ψ are dense in $\mathcal{H}^k_{\mathbb{C},\theta}$, we lose no real loss of generality by tacitly doing so.) By Theorem 4.3, $\Psi^\theta_\lambda(A) := \Psi^\theta_\lambda(\gamma^A_k)$ converges in distribution to the Ψ^A_k of (5.6), and, indeed, does so as a process over $A \in \mathcal{A}$.

The first thing we note is that the measures γ^A_k now become the natural parameters of the problem, and so one can see where the inverse Fourier transforms of (5.8) and (5.9) might come from. In fact, even (5.7) could be written in terms of inverse Fourier transforms, since it is clearly equivalent to

$$F^{-1}\overline{\pi}_k(A_1 \cup A_2) = F^{-1}\overline{\pi}(A_1) + F^{-1}\overline{\pi}_k(A_2). \tag{5.10}$$

Secondly, since $F^{-1}\overline{\pi}_k(A) \equiv \gamma^A_k$, and the γ^A_k and $F^\theta_{\gamma^A_k}$ are easy to understand, some intuition may now be forthcoming. We develop this intuition by studying additivity, locality, and stationarity not for the functional Ψ^A_k of the Gaussian field, but for the functional $\Psi^\theta_\lambda(\gamma^A_k)$ of the Markov processes which ultimately converges to Ψ^A_k. Note that we cannot rederive the results of Dobrushin and Kel'bert this way, but can only provide a way of understanding them.

a. Additivity. It is immediate from the definition of $\Psi^\theta_\lambda(\gamma^A_k)$ and $F^\theta_{\gamma^A_k}$ that the former is additive. In fact, our construction has always been such that we can build *only* additive functionals, since γ^A_k is required, *a priori*, to be a measure. Condition (5.7) (or (5.10)) simply allows our construction to be started.

b. Locality. Here we shall have to define a slightly amended definition of locality more suited to the Markov path setting. For any open $O \subset \mathbb{R}^d$, we let $\mathcal{B}_O$ denote the σ-algebra generated by the X_t while they take values in O. This notion, which we shall call the *excursion σ-field*, can be made precise. (See, for example Jacobs (1978), Kaspi (1985)). We now define locality as in §3d, but with respect to these σ-algebras.

Assume (5.8), i.e. supp $\gamma^A_k \subset \text{diag}A^k$. It is then clear from the definition of the $F^\theta_{\gamma^A_k}$ that they, and so $\Psi^\theta_\lambda(\gamma^A_k)$, are local in the new sense. Furthermore, it is easy to see that

this follows from supp $\gamma_k^A \subseteq A^k$, the additional restriction to the diagonal being seemingly superfluous. This restriction arises in a natural fashion from the additivity of Ψ_λ^θ, as in Dobrushin and Kel'bert (1983a). Using, then, what we know about functionals of Markov processes, we see that the diagonality condition implies that the k-th order functional of the Gaussian field is approximated by a functional built on the intersections of k particles at a time. This is the full import of (5.8).

Thus, with sufficient motivation, we could build an approximation to the whole of Fock space, the k-th part of this based on functionals of k particles. The reader familiar with quantum field theory will see an extremely pleasant semantic coincidence here, for the quantum theory of k interacting particles is also the theory of the k-th part of Fock space. Needless to say, our particles and those of quantum field theory have no physical link semantically however, the coincidence is intriguing.

c. Stationarity. From their construction the functionals $\Psi_\lambda^\theta(\gamma_k^A)$ are stationary in the sense that

$$\hat{U}_x \Psi_\lambda^\theta(\gamma_k^A) = \Psi_\lambda^\theta(U_x \gamma_k^A). \tag{5.11}$$

where $\hat{U}_x$ is the unitary translation operator on the $\mathcal{L}^2$ space of the Markov processes corresponding to shifting all starting points of the paths by $-x$, and U_x is defined by

$$\int_B U_x \gamma(dx_1, \ldots, dx_k) = \int_{B-x} \gamma(dx_1, \ldots, dx_k).$$

Consequently, if we want them to be stationary in the sense of Dobrushin and Kel'bert viz.

$$\hat{U}_x \Psi_\lambda^\theta(\gamma_k^A) = \Psi_\lambda^\theta(\gamma_k^{A+x}),$$

(c.f. §3d) then we require that $\gamma_k^{A+x} = U_x \gamma_k^A$. But this is precisely condition (5.9). Thus we see that this condition, too, arises as a condition on the paths of our Markov particles.

5c. The Markov property for Gaussian fields.

One of the most interesting, and most deeply studied properties of Gaussian fields is that of their Markovianess. Unlike the case for regular processes in univariate time, the

Markov property for measure of function indexed fields is not particularly useful. Nevertheless, it is probabilistically intriguing. Furthermore, it is of considerable importance in Physics, since in the transformation of regular field theory to Euclidean field theory (i.e. to Gaussian fields) one of the axioms of the former transforms to an assumption of Markovianess, (c.f. Nelson (1973)). Consequently, it is only the Markov Gaussian fields that are of interest to physicists.

There are many different definitions of Markovianess, and they are not all equivalent. Albeverio and Høegh-Krohn (1984) give a detailed discussion, but we shall follow the treatment of Dynkin (1980). Firstly, we restrict our attention to measure-indexed ($\mathcal{M}^{\theta,1}$) fields. We say that a field Φ is *Markov with respect to two sets* $A, B \subset \mathbf{R}^d$ if the σ-algebras $\mathcal{F}_{\Phi,A}$ and $\mathcal{F}_{\Phi,B}$ are conditionally independent given $\mathcal{F}_{\Phi,A\cap B}$, where $\mathcal{F}_{\Phi,A} := \sigma\{\Phi(\mu) : \text{supp } \mu \subseteq A\}$. (Note that $\mathcal{F}_{\Phi,A}$ is ***not*** the same as the $\mathcal{B}_{\Phi,A}$ of §3d.) One easily extends this to a general Markov property by demanding that for all $B \subset \mathbf{R}^d$, with smooth enough boundary ∂B and complement B^c, Φ is Markov with respect to B and B^c. It is clear therefore why it is more natural to work with measure indexed fields - $\mathcal{F}_{\Phi,\partial B}$ would be empty in the function indexed case, for we could generally find no $f \in S_d$ with support in ∂B. (A way around this is to base the Markov property on $\mathcal{B}_{\Phi,A}$ instead of $\mathcal{F}_{\Phi,A}$. This leads to a different theory which, when specialised to simple processes on $\mathbf{R}^1$, for example, is not the usual Markov theory, c.f. Rozanov (1982), Wong and Zakai (1986).)

If we limit our attention, as usual, to Gaussian fields with covariance kernel the Green function of a Markov process, then Dynkin (1980) has a very elegant characterisation of Markovianess.

Characterisation. *Φ has the Markov property on all sets A, B for which it is impossible for the associated Markov process to reach A from B without crossing $A \cap B$.* (This result also has a more precise statement - see both Dynkin (1980) and Atkinson (1983).)

We would like to understand this result from our viewpoint. In fact, it will suffice if we can see why the condition of the conjecture implies the Φ^n_λ are Markovian. This will not, of course, imply the Markovianess of Φ itself, nor can it ever hope to (the convergence

of Φ^{θ}_{λ} to Φ is in distribution only, and the Markov property is in essence an a.s. statement) but it does provide insight.

Fix λ, and, for the sake of intuition, fix $N(\lambda) = \lambda = n$ in the sum (4.3) defining $\Phi^{\theta}_{\lambda}(\mu)$, which now becomes

$$\Phi^{\theta}_{n}(\mu) = (\theta/n)^{1/2} \sum_{i=1}^{n} \sigma_i F^{\eta}_{\mu}(X_i). \tag{5.12}$$

Now we argue more or less heuristically. If the X_i have regular enough sample paths, then we claim that $\mathcal{F}(\Phi^{\theta}_{n}, A)$ is identical to the excursion σ-algebra generated by $\mathbf{X} = (X_1, \ldots, X_n)$ while the X_i are in A. Then, from rigorous results of Jacobs (1978) and Kaspi (1985) one can see that, again, requiring regular paths, $\mathcal{F}(\Phi^{\theta}_{n}, A)$ and $\mathcal{F}(\Phi^{\theta}_{n}, A^c)$ will be conditionally independent given $\mathcal{F}(\Phi^{\theta}_{n}, \partial A)$. But to apply these results the balance between regularity of the sample paths of the X_i and ∂A must be such that, at very least, one can define an additive functional of $(X_1, \ldots, X_n)$ living on ∂A. Given the other assumptions that we have placed on the X_i, this means that we require the X_i to be continuous, and so we have a heuristic version of Dynkin's characterisation result.

Actually, all of this is easiest to see when $d = 2, n = 1$, and we draw a picture.

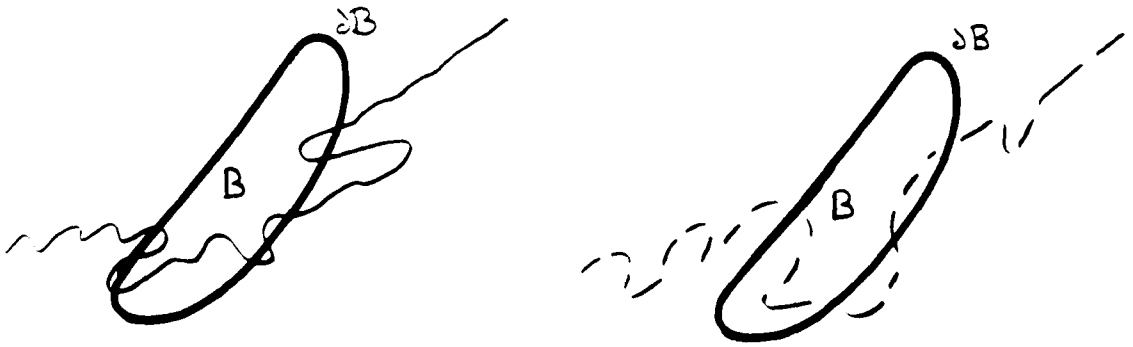

Figure 5.1. Markov paths.

In the leftmost case example of Figure 5.1 the Markov path is continuous, and it is clear that, given all the hitting points of the path on ∂B, each excursion, between hitting points, is independent of all the others. Consequently, all the excursions contained in B

are conditionally independent of those in B^c, and from this follows the Markov property of Φ_1^θ. In the rightmost case, the path may never hit ∂B, and so there is no information in $\mathcal{F}_{\Phi,\partial B}$ to even attempt to start conditioning.

Finally, we note that it is tempting to try to push these arguments further, to obtain an expression for $E\{\Phi_n^\theta(\mu) \mid \mathcal{F}_{\Phi_n^\theta,\partial B}\}$ with supp $\mu \in B^c$, as Dynkin (1980) has done, in terms of the first hitting time of X_t on ∂B, conditioning on starting X_t according to μ. It is clear how the excursion processes associated with X_t, along with time reversibility, need to be used here, but the calculations involved seem so complex that nothing is gained via this approach.

5d. The free field again.

It is instructive to close this section with an almost trivial comment on the free field of (3.5), which holds a uniquely central role in both Euclidean field theory and Gaussian fields, as the only stationary Markov field. Our question is – what, from our point of view, makes this process so special? We know that to build a Gaussian field we need Markov paths. If we require the field to be stationary then, as we have already noted, the Green function must satisfy $g^\theta(x, y) = g^\theta(x - y)$. This, however, forces the Markov processes to have stationary, independent increments. If we furthermore demand Markovianess for the field, then we demand continuity for the paths of the Markov particles. There is only one continuous process with stationary, independent increments – Brownian motion. Thus we have only one way to build the field from Markov paths. The resulting field, as we have already seen, is the free field.

6. Applications II: Renormalisation

In the previous section we used the limit theorems of Section 4 primarily at a heuristic level to explain a number of existing results. In this section we shall use these theorems in a fully rigourous fashion to motivate, and then prove, new results. The central theme of this section comes from the idea of renormalisation techniques, which we shall define formally soon.

6a. Self-similarity for fields.

In Probability Theory, renormalisation is generally (but incorrectly) considered to be synonymous with the notion of self-similarity. To make our lives notationally easier, we shall restrict our attention in this section to random fields on S_d, in which case we call a field Φ *self-similar of order* α iff for every $\eta > 0$ and $f \in S_d$

$$\Phi(f) \stackrel{\mathcal{L}}{=} {}^{\alpha}_{\eta}\Phi(f) := \Phi({}^{\alpha}_{\eta}f) \tag{6.1}$$

where

$${}^{\alpha}_{\eta}f(x) := \eta^{-\alpha} f(x/\eta). \tag{6.2}$$

This is clearly a scaling phenomenon, and in the Gaussian situation Dobrushin (1979) has shown

Theorem 6.1. *Let Φ be a stationary, zero mean, Gaussian random field with spectral measure G. Then Φ is self-similar of order α iff*

$$G(A) = \eta^{-2\alpha} G(\eta A), \qquad A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\}),\ \eta > 0. \tag{6.3}$$

As neat as this result is, there are two problems with it. Firstly, it applies only to stationary fields. Since both the statement and proof rely on spectral methods, there is no clear extension to the non-stationary case. More importantly, however, is that it deals with far too narrow a notion of self-similarity. For example, if we take Φ^θ to be the free field with mass $\theta = m^2$, and $d \geq 2$, then by (3.5) we have that

$$\begin{aligned} \eta^{-2\alpha} G(\eta A) &= \eta^{-2\alpha} (2\pi)^{-d} \int_{\eta A} \frac{d\lambda}{||\lambda||^2 + \theta} \\ &= \eta^{-2\alpha+d-2} (2\pi)^{-d} \int_{A} \frac{d\lambda}{||\lambda||^2 + \theta/\eta^2} \end{aligned}$$

after the obvious change of variable. If this is to be equal to $G(A)$, so that by Theorem 6.1 Φ^θ will be self-similar, we require $\alpha = \frac{1}{2}d - 1$, and $\theta = 0$. The condition on θ is unpleasant, for it implies that only the massless free field is self-similar.

From the point of view of Physics this is undesirable, since it is well known that *all* free fields are renormalisable (self-similar) as long as one renormalises mass at the same time as renormalising space. That is, we must be allowed to change θ in some fashion. In order to see how to do this, it is most natural to leave the spectral setting, and think of Gaussian fields as arising from sums of Markov paths. Thus for the remainder of this section, we treat only Gaussian fields whose covariance function is also an appropriate Green function. We start by looking at the structure of the Markov paths themselves.

6b. Self-similarity for Markov processes.

Let X, as usual, be a Markov process with initial measure $m(dx)$ = Lebesgue measure on $\mathbf{R}^d$. We call X ***self-similar with index*** β if for every $\eta > 0$, X and the process

$$(6.4) \qquad {}^{\beta}_{\eta}X(t) := \eta \hat{X}(\eta^{-\beta} t)$$

are identical in distribution, where $\hat{X}$ is a process (on another probability space) which has the same transition probabilities as X but initial measure $\eta^\alpha m(dx)$.

Note how we have had to change the initial distribution in order to preserve the distribution of starting points after scaling $\hat{X}$ by the factor η. If we were, for example, to start all our processes from zero no such condition would arise. (See Lamperti (1972) for a theory of self-similar Markov processes under such a condition.)

Lemma 6.1. *X has transition density $p(t; x, y)$ satisfying*

$$(6.5) \qquad p(\eta^{-\beta} t; x, y) = \eta^d p(t; \eta x, \eta y) \quad \text{for all} \quad \eta > 0, t \geq 0,$$

iff it is self-similar of index β.

Proof.

(i) **Sufficiency:** For $A \subset \mathcal{B}^d$,

$$P^{\eta^d m}(\eta \hat{X}(\eta^{-\beta} t) \in A)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \eta^d dx \int_{A/\eta} dy p(\eta^{-\beta} t; x, y) \\
&= \int_{\mathbb{R}^d} \eta^d dx \int_{A/\eta} dy \eta^d p(t; \eta x, \eta y) \qquad \text{by (6.5)} \\
&= \int_{\mathbb{R}^d} du \int_A dv p(t; u, v) \\
&= P^m(X(t) \in A).
\end{aligned}
$$

(ii) **Necessity:** By self-similarity

$$P\{\eta \hat{X}(\eta^{-\beta} t) \in A \mid \eta \hat{X}(\eta^{-\beta} t_0) = x\} = P\{X(t) \in A \mid X(t_0) = x\}.$$

i.e.

$$P_{\eta^{-\beta}(t-t_0)}(x\eta^{-1}, \eta^{-1} A) = P_{t-t_0}(x, A),$$

where $P_t(x, A)$ is the obvious transition probability function. Setting $t - t_0 = s$ and transforming the last equality to a statement about densities gives

$$p(\eta^{-\beta} s; x, y) = \eta^d p(s; \eta x, \eta y)$$

as required.

Lemma 6.2. *Let g^θ be the Green function of X (and so also of $\hat{X}$). If X is self-similar with parameter β then for all $\eta > 0$*

$$g^\theta(x, y) = \eta^{d-\beta} g^{\theta\eta^{-\beta}}(\eta x, \eta y). \tag{6.6}$$

Proof.

$$
\begin{aligned}
g^{\theta\eta^{-\beta}}(\eta x, \eta y) &= \int_0^\infty p(t; \eta x, \eta y) \exp(-t\theta\eta^{-\beta}) dt \\
&= \eta^\beta \int_0^\infty e^{-\theta s} p(s\eta^\beta; \eta x, \eta y) ds \\
&= \eta^{\beta-d} \int_0^\infty e^{-\theta s} p(s; x, y) ds \qquad \text{by (6.5)} \\
&= \eta^{\beta-d} g^\theta(x, y),
\end{aligned}
$$

from which (6.6) follows trivially.

As an example is probably in order at this point, the reader might like to check for himself that Brownian motion is self-similar with $\beta = 2$. Relation (6.6) is most easily verified from the Fourier transform representation (3.5) of the Green function.

Note that whereas Lemma 6.1 gives a necessary and sufficient condition for self-similarity, (6.6) is, as it stands, only necessary. If, however, we assume (6.6) for *all* $\theta > 0$, then it is easy to see that is also sufficient.

6c. A prologue to renormalisation.

The above shows us that many Markov processes are self similar. It thus seems reasonable that if we sum appropriate functionals of these processes then the limiting sums – that is, the Gaussian fields and their functionals – should somehow inherit this self-similarity. There is, however, one more parameter in this picture that requires normalising. Inherent in our construction of Markov functionals was an exponential damping factor. In constructing ${}^{\beta}_{\eta}X$ from X, the time change is compensated for via a change of initial measure and a scaling factor. In building functionals on ${}^{\beta}_{\eta}X$ that will be distributionally equivalent to those on X, we shall also have to compensate in the damping factor for the different time rates.

Another way to think of this is to recall, as we noted in §2b, that the exponential damping is essentially equivalent to exponential killing. From this viewpoint it is clear that when we change the time scale to obtain ${}^{\beta}_{\eta}X$ from X, we must also change the rate at which we kill ${}^{\beta}_{\eta}X$.

We need one more result before we can continue, that relates the distributions of functionals of X to those of ${}^{\beta}_{\eta}X$. As one would expect, for the right sort of functionals, self-similarity is available.

Theorem 6.2. *Take X a right Markov process with symmetric Green function g^{θ}. Let $\gamma \in \mathcal{M}^{\theta,k}$, $k \geq 1$, and define ${}^{\alpha}_{\eta}\gamma = \gamma(\alpha, \eta)$ by*

$$ {}^{\alpha}_{\eta}\gamma(A) = \eta^{-\alpha+dk}\gamma(\eta^{-1}A). \tag{6.7}$$

Then, if X is self-similar with index β,

$$F^{\theta}_{\gamma}(X_1,\ldots,X_k)\stackrel{\mathcal{L}}{=}\eta^{-k(\beta+d)/2}F^{\theta\eta^{-\beta}}_{\gamma(\alpha,\eta)}(X_1,\ldots,X_k), \qquad \alpha = k(d+\beta)/2.$$

Proof. For notational convenience, we assume that γ is absolutely continuous with density $q(\mathbf{x})$ on $\mathbf{R}^{dk}$. The general case is handled, as usual, by passage to the limit. Then (6.7) becomes, with $\alpha = k(d+\beta)/2$,

$$(6.8) \qquad {}^{\alpha}_{\eta}q(\mathbf{x}) = \eta^{-k(d+\beta)/2}q(\mathbf{x}/\eta)$$

The first thing we must check is that ${}^{\alpha}_{\eta}q \in \mathcal{M}^{\theta\eta^{-\beta},k}$; i.e. $\langle {}^{\alpha}_{\eta}q, {}^{\alpha}_{\eta}q\rangle_{\theta\eta^{-\beta}} < \infty$. But, by (2.10),

$$\begin{aligned}
\langle {}^{\alpha}_{\eta}q, {}^{\alpha}_{\eta}q\rangle_{\theta\eta^{-\beta}} &= \int {}^{\alpha}_{\eta}q(\mathbf{x})g^{\theta\eta^{-\beta}}(\mathbf{x},\mathbf{y}){}^{\alpha}_{\eta}q(\mathbf{y})d\mathbf{y} \\
&= \eta^{-k(d+\beta)}\int q(\mathbf{x}/\eta)g^{\theta\eta^{-\beta}}(\mathbf{x},\mathbf{y})q(\mathbf{y}/\eta)d\mathbf{x}d\mathbf{y} \quad \text{by (6.8)} \\
&= \eta^{k(d-\beta)}\int q(\mathbf{u})g^{\theta\eta^{-\beta}}(\eta\mathbf{u},\eta\mathbf{v})q(\mathbf{v})d\mathbf{u}d\mathbf{v} \qquad (6.9) \\
&= \int q(\mathbf{u})g^{\theta}(\mathbf{u},\mathbf{v})q(\mathbf{v})d\mathbf{u}d\mathbf{v} \qquad \text{by (6.6)} \\
&= \langle q,q\rangle_{\theta},
\end{aligned}$$

which is finite by assumption. Thus $F^{\theta\eta^{-\beta}}_{\gamma(\beta,\eta)}$ is certainly well-defined.

The next thing we shall need to know is what happens to $F^{\theta}_{\gamma}(X_1,\ldots,X_k)$ when we change the initial distribution of the X_i from $m(dx)$ to $\eta^d m(dx)$. But this is clear, for the characterisation (2.11) between F^{θ}_{γ} and γ gives us that

$$\begin{aligned}
(6.10) \qquad &\{F^{\theta}_{\gamma}(X_1,\ldots,X_k), \text{ with initial distribution } m(dx)\} \\
&\stackrel{\mathcal{L}}{=}\{F^{\theta}_{\eta^{-dk}\gamma}(X_1,\ldots,X_k) \quad \text{with initial distribution} \quad \eta^d m(dx)\}.
\end{aligned}$$

This is all the background we need. Now note

$$\begin{aligned}
F^{\theta\eta^{-\beta}}_{\gamma(\alpha,\eta)}(X_1,\ldots,X_k) &= \int_{\mathbf{R}^k_+} e^{-\theta\eta^{-\beta}|\mathbf{t}|} \quad {}^{\alpha}_{\eta}q(X_1(t_1),\ldots,X_k(t_k))dt_1\ldots dt_k \\
&= \eta^{-k(d+\beta)/2}\int_{\mathbf{R}^k_+} e^{-\theta\eta^{-\beta}|\mathbf{t}|}q(\eta^{-1}X_1(t_1),\ldots,\eta^{-1}X_k(t_k))dt_1\ldots dt_k \\
&\stackrel{\mathcal{L}}{=}\eta^{-k(\beta-d)/2}\int_{\mathbf{R}^k_+} e^{-\theta\eta^{-\beta}|\mathbf{t}|}q(\eta^{-1}\hat{X}_1(t_1),\ldots,\eta^{-1}\hat{X}_k(t_k))dt_1\ldots dt_k,
\end{aligned}$$

where the $\hat{X}_t$ have initial measure $\eta^{-d}m(dx)$, and we have applied (6.10). Map $t_i \to \eta^{\beta}\tau_i$ in the last integral to obtain

$$F^{\theta\eta^{\beta}}_{\gamma(\alpha,\eta)}(X_1,\ldots,X_k) \stackrel{\mathcal{L}}{=} \eta^{k(\beta+d)/2}\int_{\mathbf{R}^k_+} e^{-\theta|\tau|}q(\eta^{-1}\hat{X}_1(\eta^{\beta}\tau_1),\ldots,\eta^{-1}\hat{X}_k(\eta^{\beta}\tau_k))d\tau_1\ldots d\tau_k$$

$$\stackrel{\mathcal{L}}{=} \eta^{k(\beta+d)/2}F^{\theta}_{\gamma}(X_1,\ldots,X_k)$$

by self-similarity via (6.4). This proves the theorem.

We are now in a position to return to the study of Gaussian fields.

6d. Renormalising Gaussian fields.

The basic renormalisation of a function indexed field Φ is the ${}^{\alpha}_{\eta}\Phi$ given by (6.1). As we noted earlier, if $\Phi \stackrel{\mathcal{L}}{=} {}^{\alpha}_{\eta}\Phi$, Φ is called self-similar. We wish to extend the notion of self-similarity, by starting with a family of covariance kernels $g^{\theta}(x,y), \theta > 0$, which are the Green functions of a Markov process. For each θ, Φ^{θ} will be the corresponding Gaussian field. We call the family $\{\Phi^{\theta}, \theta > 0\}$ ***renormalisable*** with renormalisation parameters (α, r) if

$$\Phi^{\theta} \stackrel{\mathcal{L}}{=} {}^{\alpha}_{\eta}\Phi^{\theta\eta^{-r}}. \tag{6.11}$$

If the Φ^{θ} are measure indexed, set

$${}^{\alpha}_{\eta}\Phi(\gamma) = \Phi({}^{\alpha}_{\eta}\gamma), \tag{6.12}$$

where ${}^{\alpha}_{\eta}\gamma$ is defined at (6.7), in order to define the renormalisation parameters.

This notion extends that of self-similarity in two directions. Firstly, the two fields need not be defined on the same probability space. Secondly, there is an extra parameter available in the renormalisation in order to match the fields. This corresponds to ***mass renormalisation*** in quantum field theory (c.f. Glimm and Jaffe (1982)), and is completely missing in the usual theory of self-similarity. We have

Theorem 6.3. *Let $g^{\theta}, \theta > 0$, be a Green function of a self similar Markov process, with index β, and let $\{\Phi^{\theta}, \theta > 0\}$ be a family of Gaussian fields with covariance kernels $\{g^{\theta}, \theta > 0\}$. Then the Φ^{θ} are renormalisable with parameters $((d+\beta)/2, \beta)$.*

Proof. For this result we do not need to explicitly consider the Markov process corresponding to g^θ. In fact, since all fields are mean zero and Gaussian, we need only check that covariances match. Thus, take $f, h \in S_d$, set $\alpha = (\beta + d)/2$ and note

$$\begin{aligned}
&E\{{}^{\alpha}_{\eta}\Phi^{\theta\eta^{-\beta}}(f).{}^{\alpha}_{\eta}\Phi^{\theta\eta^{-\beta}}(h)\} \\
=&E\{\Phi^{\theta\eta^{-\beta}}({}^{\alpha}_{\eta}f).\Phi^{\theta\eta^{-\beta}}({}^{\alpha}_{\eta}h)\} \\
&= \int\int {}^{\alpha}_{\eta}f(x).g^{\theta\eta^{-\beta}}(x,y).{}^{\alpha}_{\eta}h(y)dxdy \\
&= \eta^{-2\alpha+\beta-d}\int\int f(x/\eta)g^{\theta}(x/\eta, y/\eta)h(y/\eta)dxdy \qquad \text{by (6.2), (6.6)} \\
&= \int\int f(u)g^{\theta}(u,v)h(v)dudv \\
&= E\{\Phi^{\theta}(f).\Phi^{\theta}(h)\}.
\end{aligned}$$

Thus the covariances do, in fact, match, and the theorem is proven.

In the final result of this section we consider the renormalisability of additive functionals of Gaussian fields. In essence, we have already done this, for we know via the specific results of §4 and the more general results of §8 that all additive functionals of Gaussian fields can be approximated by sums of additive functionals (c.f. (4.5)) and Theorem 6.2 tells us when the latter are renormalisable.

For completeness, however, let us state the following result for the stationary case, for which the reader does not yet have to know the results of §8.

Theorem 6.4. *Under the assumptions of Theorem 6.3, take a* $\gamma \in {}^{*}\mathbf{M}^{\theta,k}_{symm}$. *Define for* $\eta > 0$, ${}^{\alpha}_{\eta}\gamma$ *as in (6.7). Let* $\Psi^{\theta}(\gamma)$ *and* $\Psi^{\theta\eta^{-\beta}}({}^{\alpha}_{\eta}\gamma)$ *be as in (4.4). Then*

$$\Psi^{\theta}(\gamma) \stackrel{\mathcal{L}}{=} \Psi^{\theta\eta^{-\beta}}({}^{\alpha}_{\eta}\gamma), \qquad \alpha = k(d+\beta)/2. \tag{6.13}$$

Proof. The result follows trivially from Theorems 4.3 and 6.2. We need only check that ${}^{\alpha}_{\eta}\gamma \in {}^{*}\mathbf{M}^{\theta\eta^{-\beta},k}_{symm}$. But since γ is symmetric with inverse Fourier transform of the right kind, it is automatically in ${}^{\alpha}_{\eta}\gamma$. Thus only integrability needs to be checked. This, however, is the calculation at (6.9), so we are done.

In the terminology of normalisation parameters, we have

Corollary. $\Psi^{\theta}(\gamma)$ *is renormalisable for the pair* $(k(d+3)/2, 3)$.

Proof. Compare the definition (6.12), with the result (6.13) and the definition (6.7) of the renormalised γ.

7. Moment Proofs

In this section we shall give a proof of the limit theorems of Section 4 via the method of moments. For reasons we shall outline below, this methodology works perfectly for Theorems 4.1 and 4.2, which treat only convergence to the Gaussian limit, but breaks down in Theorem 4.3 if the integral (4.4) defining the functional Ψ^θ_γ is more than two-dimensional. (Or, equivalently, if the F^θ_γ of (4.5) is defined on more than two X_t at a time.)

In essence, the proofs of this section are redundant, for more powerful techniques, which work in more general situations, will be used in the following section. Nevertheless, we include them for two reasons: firstly, they are conceptually easier and considerably more concrete than those of Section 8, and, secondly, we find it both interesting and illuminating to calculate the moments of the limiting distribution *per se*.

To make life a little easier, we note the obvious fact (e.g. Reed and Simon (1972), p. 51) that linear combinations of functions of the form

$$f_o(x_1,\ldots,x_k) = f(x_1)\ldots f(x_k) \qquad f \in \mathcal{H}^1_{G^\theta} \tag{7.1}$$

are dense in $\mathcal{H}^k_{G^\theta}$, and linear combinations of measures of the form

$$\gamma_o(dx_1,\ldots,dx_k) = \gamma(dx_1)\ldots\gamma(dx_k) \qquad \gamma \in \mathcal{M}^{\theta,1}_{symm} \tag{7.2}$$

are dense in $\mathcal{M}^{\theta,k}_{symm}$. Consequently, since the functionals $\Phi^\theta(\cdot)$, $\Phi^\theta_\lambda(\cdot)$, $\Psi^\theta(\cdot)$ and $\Psi^\theta_\lambda(\cdot)$ are all continuous in probability, it clearly suffices to prove the Theorems of Section 4 for f_o and γ_o of the form (7.1) and (7.2) respectively.

In this section we shall limit our discussion even further, to the case of $k = 2$, because of the following result.

Lemma 7.1. *For $f \in S_d$ and $\gamma \in {}^*\mathcal{M}^{\theta,2}_{symm}$, the moments of $\alpha\Phi^\theta(f) + \beta\Psi^\theta(\gamma)$, $\alpha, \beta \in \mathbf{R}$, determine the joint distribution of $\Phi^\theta(f)$ and $\Psi^\theta(\gamma)$.*

Proof. By Feller (1979), Problem 30.3, for example, the joint distribution is determined

by the mixed moments if, for some $\sigma > 0$,

$$\overline{\lim}_{n\to\infty} \frac{E\{|X|^n\}}{\sigma^n n!} < \infty, \tag{7.3}$$

with $X = \Phi^\theta(f)$ or $\Psi^\theta(\gamma)$. But standard results on multiple Wiener-Ito integrals (e.g. Major (1981), p. 119) tell us that the n-th moment of a k-fold integral is of order $\exp[(k/2)n\log n + O(n)]$. Since Φ^θ and Ψ^θ correspond to $k = 1$ and $k = 2$, condition (7.3) is satisfied. Note, however, that for $k > 2$ this condition is not satisfied, and, indeed, moment methods do not work in this situation.

We now turn to calculating the mixed moments of the lemma, for which we shall need certain *Feynman diagrams*. The diagrams that will suffice for our purposes consist of a set of *vertices*, labelled $1,\ldots,m$, and *legs* labelled (i,j), $j = 1,\ldots,k_i$, which belong to the i-th vertex. We shall require only the cases $k_i = 1$ or 2. A diagram, denoted by $(k_1,\ldots,k_m)$ is formed by pairing all the legs in such a way that the legs in each pair belong to different vertices. Figure 7.1 shows the two unique (up to permutations of legs belonging to a given vertex, or of like vertices among themselves) diagrams for $(1,1,2,2)$. (Note, that in the notation of Major (1981) and Dobrushin (1979) we consider only what they call the *complete* diagrams.) Note that in general there are $K = k_1 + \ldots + k_m$ legs, and so $K/2$ pairs.

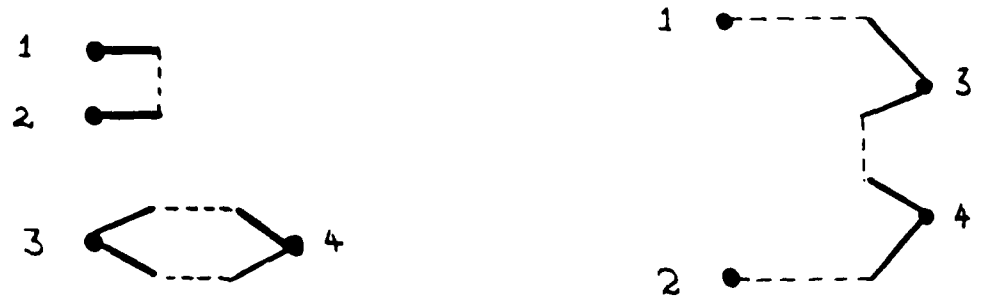

Figure 7.1. The two diagrams for $(1,1,2,2)$.
The broken lines represent bonds, the heavy lines represent legs.

We can now turn to the moments of $\alpha\Phi^\theta + \beta\Psi^\theta$. Clearly, it will suffice to find an expression for

$$
\begin{aligned}
&E\{[\Phi^\theta(f)]^m[\Psi^\theta(\gamma)]^{n-m}\} \\
&=E\{[\int \hat{f}(\lambda)Z_{G^\theta}(d\lambda)]^m\cdot[\tfrac{1}{2}\int\int \hat{\gamma}(\lambda_1)\hat{\gamma}(\lambda_2)Z_{G^\theta}(d\lambda_1)Z_{G^\theta}(d\lambda_2)]^{n-m}\},
\end{aligned} \tag{7.4}
$$

for $n = 1, 2, \ldots$, $m = 0, 1, \ldots, n$. But this is now easy, for prepared formulae for the expectation of moments of Wiener-Ito integrals already exist. To present them, let $\Gamma(n, m)$ be the collection of all n-vertex Feynman diagrams for which the first m vertices have only one leg, and the remaining $n - m$ vertices two legs each. Since there are $m + 2(n - m) = 2n - m$ legs, and so $n - \frac{1}{2}m$ bonds, m must be even.

We number the legs by an index α, $\alpha = 1, \ldots, 2n - m$, and denote a bond by a pair (α_1, α_2). (Clearly, not all 2-tuples are possible.) Let A be the set of α for which the α-th leg comes from a 2-leg vertex. Define

$$
r_{\alpha\beta} = \begin{cases} \int |\hat{f}(\lambda)|^2 G^\theta(d\lambda) & \alpha, \beta \in A^c, \\ \int |\hat{\gamma}(\lambda)|^2 G^\theta(d\lambda) & \alpha, \beta \in A, \\ \int \hat{\gamma}(\lambda)\hat{f}^*(\lambda) G^\theta(d\lambda) & \alpha \in A, \beta \in A^c \\ & \text{or } \alpha \in A^c, \beta \in A. \end{cases} \tag{7.5}
$$

Note that in the third case the various symmetry requirements give us that $r_{\alpha\beta}$ is also given by

$$
\int \hat{\gamma}(\lambda)\hat{f}(-\lambda)G^\theta(d\lambda) = \int \hat{\gamma}(-\lambda)\hat{f}(\lambda)G^\theta(d\lambda). \tag{7.6}
$$

We can now state

Lemma 7.2. *For $f \in S_d$ and $\gamma \in {}^*\mathcal{M}^{\theta,2}_{symm}$, $n \geq 1$, $m = 0, 2, \ldots, 2[n/2]$,*

$$
E\{[\Phi^\theta(f)]^m[\Psi^\theta(\gamma)]^{n-m}\} = 2^{m-n}\Sigma r_{\alpha_1\beta_1}\ldots r_{\alpha_q\beta_q} \tag{7.7}
$$

where the sum is taken over all diagrams in $\Gamma(n, m)$, and $q = n - \frac{1}{2}m$. For m odd, the mixed moment is zero.

Proof. All the hard work has already been done, in setting up the notation. The Lemma is a straightforward special case of Corollary 5.4 of Major (1981).

Before we can finally turn to the proof of Theorem 4.3, we need some information on the moments of functionals of Markov processes.

Lemma 7.3. *Let X be a Markov process as in Section 2, and $\gamma_1, \gamma_2 \in \mathbf{M}^{\theta,1}$. Then*

$$E\{F^\theta_{\gamma_1}(X)F^\theta_{\gamma_2}(X)\} = \theta^{-1}\int \gamma_1(dx)g^\theta(x,y)\gamma_2(dy) \tag{7.8}$$
$$= \theta^{-1}\langle\gamma_1,\gamma_2\rangle_\theta.$$

In particular,

$$E\{|F^\theta_\gamma(X)|^2\} = \theta^{-1}\langle\gamma,\gamma\rangle_\theta.$$

Lemma 7.4. *Let X be a Markov process as in Section 2, and $\gamma \in \mathbf{M}^{\theta,1}$. Then, for $n \geq 1$,*

$$E\{|F^\theta_\gamma(X)|^n\} = \frac{(n-1)!}{\theta}\int\cdots\int g^{(n-1)\theta}(x_1,x_2)\cdots g^{1\theta}(x_{n-1},x_n)\gamma(dx_1)\cdots\gamma(dx_n)$$
$$\leq \frac{(n-1)!}{\theta}\int\cdots\int g^{\theta}(x_1,x_2)\cdots g^{\theta}(x_{n-1},x_n)\gamma(dx_1)\cdots\gamma(dx_n). \tag{7.9}$$

Both of these lemmas are of the same form as Theorem 5.2 of Dynkin (1984), where similar moments are calculated, albeit for a slightly different class of processes and functionals. It is a straightforward, although tedious, exercise to follow through Dynkin's calculations to obtain the lemmas.

We shall also require

Lemma 7.5. *Let X be a Markov process as in Section 2, and $\gamma_2 \in \mathbf{M}^{\theta,2}$ such that $\gamma_2(dx_1,dx_2) = \gamma_1(dx_1)\gamma_1(dx_2)$ with $\gamma_1 \in \mathbf{M}^{\theta,1}$. If $F^\theta_{\gamma_2}$ and $F^\theta_{\gamma_1}$ are the corresponding additive functionals, then*

$$F^\theta_{\gamma_2}(X_1,X_2) = F^\theta_{\gamma_1}(X_1)F^\theta_{\gamma_1}(X_2).$$

Proof. In the notation of (2.13)-(2.14), we have

$$h_2(x_1,x_2)\gamma_2'(dx_1,dx_2) = \gamma_2(dx_1,dx_2)$$
$$= \gamma_1(dx_1)\gamma_1(dx_2)$$
$$= h_1(x_1)h_1(x_2)\gamma_1'(dx_1)\gamma_1'(dx_2).$$

for some finite γ_1'. Thus, with $\delta = (\delta_1,\delta_2)$,

$$h_2(x_1,x_2)b^\theta_{\gamma_2,\delta}(x_1,x_2) = h_1(x_1)h_1(x_2)\int e^{-\theta(\delta_1+\delta_2)}p_{\delta_1}(x_1,y_1)p_{\delta_2}(x_2,y_2)\gamma_1'(dx_1)\gamma_1'(dx_2)$$
$$= h_1(x_1)b^\theta_{\gamma_1,\delta_1}(x_1).h_1(x_2)b^\theta_{\gamma_1,\delta}(x_2).$$

Applying the above equivalences to the defining form for the $F^{\theta}_{\gamma_i}$ establishes the Lemma.

We can now finally turn the

Proof of Theorem 4.3 for the case $k = 2$

Firstly, we shall need to add one more assumption to those surrounding Theorem 4.3, viz. the integrals

$$\int \dots \int g^{\theta}(x_1, x_2) \dots g^{\theta}(x_{n-1}, x_n)\gamma(dx_1) \dots \gamma(dx_n) \tag{7.10}$$

are finite for all $n \geq 1$. Since by Lemma 7.4 these integrals bound the n-th moment of the limit variable $\Psi^{\theta}(\gamma)$ of the theorem, their finiteness is a natural ingredient of any moment proof.

We commence by noting that by Lemma 7.1 we need only prove convergence of mixed moments to obtain the convergence in distribution of the Theorem. By (7.1), (7.2) and the comments there we can restrict ourselves to $\gamma_2 \in {}^*\mathcal{M}^{\theta,2}_{symm}$ which are of the form $\gamma \times \gamma$, with $\gamma \in {}^*\mathcal{M}^{\theta,1}_{symm}$.

To make life notationally easier, let us write the field $\Phi^{\theta}_{\lambda}(f)$ of (4.1) in the form $\Phi^{\theta}_{\lambda}(\gamma_1)$ of (4.3), where $\gamma_1 = \gamma_1(f)$ is defined by $\gamma_1(A) = \int_A f(x)dx$, and we have $\gamma_1 \in \mathcal{M}^{\theta,1}$. Consequently, we are interested in studying the mixed moments of $\langle \Phi^{\theta}_{\lambda}(\gamma_1), \Psi^{\theta}_{\lambda}(\gamma_2) \rangle$, with $\gamma_1 \in \mathcal{M}^{\theta,1}$ and $\gamma_2 \in {}^*\mathcal{M}^{\theta,2}_{symm}$ and of product form. We have, for $n \geq 1$, $m \leq n$, $\lambda > 0$,

$$E\{[\Phi^{\theta}_{\lambda}(\gamma_1)]^m [\Psi^{\theta}_{\lambda}(\gamma_2)]^{n-m}\}$$

$$\begin{aligned} &= E\{|(\theta/\lambda)^{1/2} \sum_{i=1}^{N(\lambda)} \sigma_i F^{\theta}_{\gamma_1}(X_i)|^m . |(\theta/\lambda) \sum\sum_{1 \leq j < k \leq N(\lambda)} \sigma_j \sigma_k F^{\theta}_{\gamma_2}(X_j, X_k)|^{n-m}\} \\ &= 2^{m-n}(\theta/\lambda)^{n-1/2m} \sum_{N=2}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} \Sigma' E\{\pi^{\sigma}(\mathbf{i},\mathbf{j},\mathbf{k})\} . E\{\pi^{F}(\mathbf{i},\mathbf{j},\mathbf{k})\}, \end{aligned} \tag{7.11}$$

where $\mathbf{i} = (i_1, \dots, i_m)$, $\mathbf{j} = (j_1, \dots, j_{n-m})$, $\mathbf{k} = (k_1, \dots, k_{n-m})$,

$$\pi^{\sigma}(\mathbf{i},\mathbf{j},\mathbf{k}) = \sigma_{i_1} \dots \sigma_{i_m} \sigma_{j_1} \dots \sigma_{j_{n-m}} \sigma_{k_1} \dots \sigma_{k_{n-m}},$$

$$\begin{aligned} \pi^{F}(\mathbf{i},\mathbf{j},\mathbf{k}) &= F^{\theta}_{\gamma_1}(X_{i_1}) \dots F^{\theta}_{\gamma_1}(X_{i_m}) F^{\theta}_{\gamma_2}(X_{j_1}, X_{k_1}) \dots F^{\theta}_{\gamma_2}(X_{j_{n-m}}, X_{k_{n-m}}) \\ &= F^{\theta}_{\gamma_1}(X_{i_1}) \dots F^{\theta}_{\gamma_1}(X_{i_m}) F^{\theta}_{\gamma}(X_{j_1}) \dots F^{\theta}_{\gamma}(X_{j_{n-m}}) F^{\theta}_{\gamma}(X_{k_1}) \dots F^{\theta}_{\gamma}(X_{k_{n-m}}). \end{aligned}$$

(the last equality following from Lemma 7.5), and the sum Σ' in (7.11) is over all $\mathbf{i},\mathbf{j},\mathbf{k}$ with components in $\{1,\ldots,N\}$ such that $j_\alpha \neq k_\alpha, \alpha = 1,\ldots,n-m$.

Consider the expectation of the product $\pi^\sigma(\mathbf{i},\mathbf{j},\mathbf{k})$. This is clearly zero, unless each σ_α appears an even number of times among the $2n-m$ σ's. Thus, if m is odd, the mixed moment is zero, as is, by Lemma 7.2, is the mixed moment of Φ^θ and Ψ^θ. Henceforth, therefore, we assume m is even.

We now consider those configurations $(\mathbf{i},\mathbf{j},\mathbf{k})$ composed only of $q = n - m/2$ different indices, $(\alpha_1,\ldots,\alpha_q)$, each α_ν appearing precisely twice. Then, clearly,

$$\pi^\sigma(\mathbf{i},\mathbf{j},\mathbf{k}) = \prod_{\nu=1}^{q} \sigma_{\alpha_\nu}^2 = 1. \tag{7.12}$$

Consider the corresponding $\pi^F(\mathbf{i},\mathbf{j},\mathbf{k})$, appearing in (7.12). To each $\sigma_{\alpha_\nu}^2$ in (7.12) we can associate a term r_{α_ν} coming from the expectation of the product of two functionals of X_{α_ν}, according to the following rules: (c.f. Lemma 7.3). If $\alpha = i_a = j_b$, or $\alpha = i_a = k_b$, then

$$r_\alpha = E\{F_{\gamma_1}^\theta(X_{\alpha_\nu})F_\gamma^\theta(X_{\alpha_\nu})\} = \theta^{-1}\langle\gamma_1,\gamma\rangle_\theta. \tag{7.13}$$

If $\alpha = i_a = i_b$, then

$$r_\alpha = E\{F_{\gamma_1}^\theta(X_{\alpha_\nu})F_{\gamma_1}^\theta(X_{\alpha_\nu})\} = \theta^{-1}\langle\gamma_1,\gamma_1\rangle_\theta. \tag{7.14}$$

If $\alpha = j_a = j_b$, or $\alpha = j_a = k_b$, or $\alpha = k_a = k_b$, then

$$r_\alpha = E\{F_\gamma^\theta(X_{\alpha_\nu})F_\gamma^\theta(X_{\alpha_\nu})\} = \theta^{-1}\langle\gamma,\gamma\rangle_\theta. \tag{7.15}$$

Consequently, still in the case of precisely q different indices, the expectations of the π^F terms in (7.11) depend only on the patterns formed among the indices, and not on the indices themselves. Now note that we could choose $(\alpha_1,\ldots,\alpha_q)$ from $(1,\ldots,N)$ in $(N!)/((N-q)!)$ different ways, so that we have

$$E\{[\Phi_\lambda^\theta(\gamma_1)]^m[\Psi_\lambda^\theta(\gamma_2)]^{n-m}\} \tag{7.16}$$

$$= 2^{m-n} \sum_{N=q}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} \frac{N!}{(N-q)!} \lambda^{-q} \Sigma' \theta^q E\{\pi^F(\mathbf{i},\mathbf{j},\mathbf{k})\} + \text{ other terms}$$

$$= 2^{m-n} \Sigma' \theta^q E\{\pi^F(\mathbf{i},\mathbf{j},\mathbf{k})\} + \text{ other terms},$$

where the sum is over all the $(\mathbf{i},\mathbf{j},\mathbf{k})$ of (7.11) of the paired form discussed above. The "other terms" come from choosing $q' < q$ different indices $(\alpha_1, \ldots, \alpha_{q'})$, so that some of the σ_i appear $2p$ times, $p > 1$. These terms are asymptotically negligible, since under (7.10) boundedness of the moments of $F^{\theta}_{\gamma_1}$ and F^{θ}_{γ} (Lemma 7.4) guarantee that the $(q-1)$ "other terms" are of order

$$\sum_{N=q'}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} \cdot \frac{N!}{(N-q')!} \lambda^{-q} = \lambda^{-(q-q')} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus it remains for us to consider the λ-independent expression

$$2^{m-n} \Sigma' \theta^q \{\pi^F(\mathbf{i},\mathbf{j},\mathbf{k})\}. \tag{7.17}$$

This is clearly expressible in terms of the r_α of (7.13)-(7.15). Now recall the initial definition of γ_1 in terms of f, to note that if G^θ is the Fourier transform of g^θ then:

$$\langle \gamma_1, \gamma_1 \rangle_\theta = \int |\hat{f}(\lambda)|^2 G^\theta(d\lambda),$$

$$\langle \gamma, \gamma \rangle_\theta = \int |\hat{\gamma}(\lambda)|^2 G^\theta(d\lambda),$$

$$\langle \gamma, \gamma_1 \rangle_\theta = \int \hat{\gamma}(\lambda) \hat{f}^*(\lambda) G^\theta(d\lambda).$$

That is, the r_α's of (7.13)-(7.15) correspond to the $r_{\alpha\beta}$'s of (7.5), modulo a factor of θ. Furthermore, we can write the sum in (7.17) as a sum over Feynman diagrams $\Gamma(n,m)$, in which the m single-leg vertices correspond to indices in $\mathbf{i}$, and the $n-m$ double-leg vertices correspond to indices in $\mathbf{j}$ and $\mathbf{k}$. Thus, since the factors of θ cancel nicely, we have that (7.17) is precisely (7.7). That is, the mixed moments of $\Phi^\theta_\lambda(f)$ and $\Psi^\theta_\lambda(\gamma_2)$ converge to those of $\Phi^\theta(f)$ and $\Psi^\theta(\gamma_2)$. This completes the proof.

A remark on $k > 2$. It is possible to show that even if $k > 2$ the moment sequence of the approximating sums converge to those of the Gaussian limit. The only essential difference in the proof is that more complicated Feynman diagrams are required. Wolpert (1978b) has actually done this for the special case of X Brownian motion in $\mathbf{R}^2$, and the additive functional of the limiting field a Wick power. Note, however, that for $k > 2$ the moments of Gaussian functionals do not determine their distributions, and so no convergence in distribution result can be obtained this way.

We conclude this section with some brief comments on the

Proofs of Theorems 4.1 and 4.2. It is clear that both of these results are virtually standard central limit theorems, so that all we need check is that the summands satisfy appropriate moment conditions and the mean and variance of the limit are correct. Both of these, however, follow from Lemma 7.3, and so the proofs are trivial.

8. The General Limit Theorem

In this, essentially final, section we shall state and prove a more general result than those of Sections 4 and 7. The motivation for, and intuitive appeal of this result is as for the earlier ones. Having seen how the arguments of Sections 5 and 6 apply to the Theorems of Section 4, the reader who has got this far will see how they also apply in the most general situation.

In order to set up this general result, we shall need a theory of multiple Wiener integrals far more general than that of Section 3. We thus recall such a theory, due to Dynkin and Mandelbaum (1983), that has its roots in Neveu (1968). We follow Mandelbaum and Taqqu (1984) most closely.

8a. Wiener integrals on an arbitrary space.

Let $(\mathfrak{X}, \mathcal{B}, \nu)$ be an arbitrary measure space, and define a Gaussian process I_1 on $\mathcal{L}^2(\nu)$ with zero mean and covariance

$$E\{I_1(f_1)I_1(f_2)\} = \nu(f_1 f_2) := \int_{\mathfrak{X}} f_1(x) f_2(x) \nu(dx). \tag{8.1}$$

If we consider the subfamily

$$\{W(B) = I_1(1_B), \quad B \in \mathcal{B}, \nu(B) < \infty\}$$

as a Wiener process on $\mathfrak{X}$, then I_1 can be written symbolically as

$$I_1(f) = \int_{\mathfrak{X}} f(x) W(dx).$$

Now write $\mathcal{H}_k = \mathcal{L}^2_{symm}(\nu \times \ldots \times \nu)$ for the space of symmetric functions $h_k(x_1, \ldots, x_k)$ for which

$$\nu^k(h_k^2) = \int h_k^2(x_1, \ldots, x_k) \nu(dx_1) \ldots \nu(dx_k) < \infty.$$

The multiple Wiener integral of order k is a *linear* mapping I_k from $\mathcal{H}_k$ into the space of random variables which are functionals of the Gaussian family $I_1(f)$. The mapping is defined uniquely by conditions A and B following:

(A) For functions of the form

$$h_k^f(x_1, \ldots, x_k) = f(x_1) \ldots f(x_k), \qquad f \in \mathcal{L}^2(\nu) \tag{8.2}$$

we have

$$I_k(h_k^f) = (\nu(f^2))^{k/2} H_k\left(\frac{I_1(f)}{\sqrt{\nu(f^2)}}\right). \tag{8.3}$$

where H_k is the Hermite polynomial of degree k with leading coefficent 1.

(B) For $h_k \in \mathcal{H}_k$,

$$E[I_k^2(h_k)] = k!\nu^k(h_k^2).$$

Again, symbolically, it makes sense to write

$$I_k(h_k) = \int_{\mathcal{X}} \ldots \int_{\mathcal{X}} h_k(x_1, \ldots, x_k) W(dx_1) \ldots W(dx_k).$$

It is important to note for later use that the linear combinations of functions satisfying (8.2) are dense in $\mathcal{H}_k$, and so their integrals are dense in the space of all integrals.

8b. A general limit theorem.

The following theorem follows from Theorem 2 and the discussion of Section 1.5 of Dynkin and Mandelbaum (1983).

Theorem 8.1. *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables taking values in $(\mathcal{X}, \mathcal{B})$, and with distribution ν. For $\lambda > 0$ let N_λ be a Poisson variable with mean λ, independent of the X_i. Let $\sigma_1, \sigma_2, \ldots$, be an i.i.d. sequence of variables with $P\{\sigma_i = +1\} = P\{\sigma_i = -1\} = \frac{1}{2}$, and independent of both the N_λ and X_i. Finally, for $k = 1, 2, \ldots$, take $h_k \in \mathcal{H}_k$ and let $I_k(h_k)$ be its multiple Wiener integral as in the previous subsection. Then the random variables*

$$Z_\lambda(h_1, h_2 \ldots) := \sum_{k=1}^{\infty} \lambda^{-k/2} \sum \sigma_{i_1} \ldots \sigma_{i_k} h_k(X_{i_1}, \ldots, X_{i_k}), \tag{8.4}$$

with the inner sum over $1 \le i_1 < i_2 < \ldots < i_k \le N_\lambda$, converge in distribution, as $\lambda \to \infty$, to

$$Z(h_1, h_2 \ldots) := \sum_{k=1}^{\infty} \frac{1}{k!} I_k(h_k). \tag{8.5}$$

As noted, an elegant proof of this powerful result can be found in Dynkin and Mandelbaum (1983). In Mandelbaum and Taqqu (1984) the result is extended to an invariance principle.

Now let us consider a special case of Theorem 8.1. Let X be a right Markov process with symmetric Green function g^θ. Let $\mathfrak{X} = (\mathbf{R}^d)^{\mathbf{R}_+}$ be the path space of X, and ν the measure the process induces on $\mathfrak{X}$. For $k \geq 1$, and $\gamma_k \in \mathcal{M}^{k,\theta}$, let $F^\theta_{\gamma_k}$ be the corresponding functional (as in §2.b) of k independent copies of X. Then, in the above notation, $F^\theta_{\gamma_k} \in \mathcal{H}_k$, and so the multiple Wiener integral $I_k(F^\theta_{\gamma_k})$ is well defined. Since there is a one-one correspondence between functionals $I_k(F^\theta_{\gamma_k})$ and the pair (γ_k, θ), we save on notation by setting

$$I_k(\gamma_k, \theta) \equiv I_k(F^\theta_{\gamma_k}). \tag{8.6}$$

Let N_λ and $\sigma_1, \sigma_2, \ldots$ be as in Theorem 8.1, and $X_1, X_2, \ldots$, i.i.d. copies of X. We have, as an immediate consequence of Theorem 8.1,

Theorem 8.2. *For* $\gamma_k \in \mathcal{M}^{\theta,k}$

$$\sum_{k=1}^{\infty} (\theta/\lambda)^{k/2} \Sigma \sigma_{i_1} \ldots \sigma_{i_k} F^\theta_{\gamma_k}(X_{i_1}, \ldots X_{i_k})$$

converges in distribution, as $\lambda \to \infty$, *to*

$$\sum_{k=1}^{\infty} \frac{\theta^{k/2}}{k!} I_k(\gamma_k, \theta).$$

This result completely incorporates the results of Section 4. and is, in fact, far more general, since far fewer restrictions are placed on the γ_k. The restrictions there, that $\gamma_k \in {}^*\mathcal{M}^{\theta,k}_{symm}$, were placed primarily to permit easy identification of the limit as a simple Wiener-Ito integral. In the more general result above, it is not quite as clear what the $I_k(\gamma_k, \theta)$ are.

To relate Theorem 8.2 with what has gone before, let us use it to obtain a new

Proof of Theorem 4.3.

We have to prove convergence of the pair $\langle \Phi^\theta_\lambda(f), \Psi^\theta_\lambda(\gamma)\rangle$ of (4.1) and (4.5) to the pair $\langle \Phi^\theta(f), \Psi^\theta(\gamma)\rangle$ of Section 3 and (4.4). The restrictions of Theorem 4.3 placed f in $S(g^\theta)$, and γ in ${}^*\mathcal{M}^{\theta,k}_{symm}$.

Firstly, to make comparisons easier, affix a suffix k to γ: i.e. $\gamma_k \equiv \gamma$. Now use $\gamma_1 = \gamma_1(f)$ to denote the measure in $\mathcal{M}^{\theta,1}$ with density f. Then, comparing (4.1) and (4.5) with the definition of an additive functional in Section 2, it is immediate that we have

$$\Phi^\theta_\lambda(f) \equiv \Psi^\theta_\lambda(\gamma_1(f)).$$

Consequently, by Theorem 8.2 and the Cramér-Wold device, we have that $\langle \Phi^\theta_\lambda(f), \Psi^\theta_\lambda(\gamma_k)\rangle$ converge in distribution to some pair $\langle \theta^{1/2} I_1(\gamma_1), \frac{\theta^{k/2}}{k!} I_k(\gamma_k, \theta)\rangle$. We need only check that this is, distributionally, the right pair.

We know that $I_1(\cdot)$ is Gaussian, with zero mean. Its covariance function is given, for $\mu_1, \mu_2 \in \mathcal{M}^{\theta,1}$, by

$$\begin{aligned} E\{I_1(\mu_1) I_1(\mu_2)\} &= E\{F^\theta_{\mu_1} F^\theta_{\mu_2}\} \\ &= \theta^{-1} \int\int \mu_1(dx) g^\theta(x,y) \mu_2(dy). \end{aligned}$$

(c.f. Lemma 7.3). Since γ_1 has density f, the last integral leads to $\mathrm{var}(I_1(\gamma_1)) = \theta^{-1} \int\int f(x) g^\theta(x,y) f(y) dx dy$, and so $I_1(\gamma_1) \stackrel{\mathcal{L}}{=} \theta^{-1/2} \Phi^\theta(f)$. To complete the proof, we consider $\gamma_k \in {}^*\mathcal{M}^{\theta,k}_{symm}$ of the form

$$\gamma_k(dx_1, \ldots, dx_k) = \gamma(dx_1) \ldots \gamma(dx_k), \quad \gamma \in {}^*\mathcal{M}^{\theta,1}_{symm}.$$

The linear combinations of such γ_k are dense in ${}^*\mathcal{M}^{\theta,k}_{symm}$, and so it will suffice to consider them. We have

$$\begin{aligned} I_k(\gamma_k, \theta) &= I_k(F^\theta_{\gamma_k}) \\ &= [E((F^\theta_\gamma)^2)]^{k/2} H_k\left(\frac{I_1(F^\theta_\gamma)}{\sqrt{E((F^\theta_\gamma)^2)}}\right). \end{aligned} \tag{8.7}$$

by (8.3). But, since the equivalence between I_1 and Φ^θ has already been established, and $\gamma \in {}^*\mathcal{M}^{\theta,1}_{symm}$ we have

$$I_1(F^\theta_\gamma) \stackrel{\mathcal{L}}{=} \theta^{-1/2} \int \hat{\gamma}(\lambda) Z_{G^\theta}(d\lambda), \tag{8.8}$$

via the Wiener-Ito integrals of Section 3. Now apply Ito's lemma (e.g. Major (1981), p. 30) to (8.7) via (8.8) to obtain

$$H_k\left(\frac{I_1(F^\theta_\gamma)}{\sqrt{E((F^\theta_\gamma)^2)}}\right) \stackrel{\mathcal{L}}{=} (\theta E((F^\theta_\gamma)^2))^{-k/2} \int \dots \int \hat{\gamma}(\lambda_1) \dots \hat{\gamma}(\lambda_k) Z_{G^\theta}(d\lambda_1) \dots Z_{G^\theta}(d\lambda_k).$$

Thus

$$\begin{aligned} \frac{\theta^{k/2}}{k!} I_k(\gamma_k, \theta) &\stackrel{\mathcal{L}}{=} \frac{1}{k!} \int \dots \int \hat{\gamma}(\lambda_1) \dots \hat{\gamma}(\lambda_k) Z_{G^\theta}(d\lambda_1) \dots Z_{G^\theta}(d\lambda_k) \\ &= \Psi^\theta(\gamma_k) \end{aligned}$$

by (4.4). Furthermore, the above argument also establishes that the joint distribution of the limit variables is the correct one, so we are done.

9. Concluding Comments.

Some comments on what we have not done:

1. The connection with Physics. Throughout the paper we have made numerous references to Euclidean quantum field theory, without ever being really precise. This was to avoid trebling (at least) the length of an already overlong paper. The reader now interested enough to find out more about this could start with Glimm and Jaffe (1982).

2. The discrete case. We have not considered lattice indexed Gaussian fields. It is clear that this could be done by representing them as the sum of random walks on the lattice, rather than the more general Markov processes we have considered. Some of these results could be obtained from ours by using the "discretization" trick of Dobrushin (1979). This, however, would be like using a sledgehammer to kill the proverbial ant, since a direct approach would be very simple. For an idea of how this might go, see Williams (1973), Spitzer (1974), or Dynkin (1983).

3. No transition density. Not every Markov process has a transition density, nor does every Gaussian field have a covariance functional that can be written via a kernel. In the more general case, Dynkin's theory of additive functionals (§2) based on the Green function and symmetry could be replaced by a theory based on the resolvent operator and duality (c.f. Dynkin (1982) and Getoor and Sharpe (1985)), and so a more general class of Gaussian fields considered. Since no new insight would be forthcoming, while a lot of very difficult Mathematics would, we shall leave this to our betters.

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